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[ LECTURE NOTES

VOLUME 5 ]

Linear operators and their spectra

by

Kurt Friedrichs

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LINEAR OPERATORS AND THEIR SPECTRA

by Kurt Friedrichs

The theory of linear operators and their spectra has emerged from generalizing the problem of principal axes of quadratic forms. Let

$$xQx = q_{11}x_1^2 + 2q_{12}x_1x_2 + q_{22}x_2^2$$

be a quadratic form,  $xQx = 1$  a quadric in the  $(x_1, x_2)$ -plane, then the problem is to find a new system of rectangular coordinates  $(a_1, a_2)$  such that

$$xQx = \frac{a_1^2}{\alpha_1^2} + \frac{a_2^2}{\alpha_2^2}$$

$$x_1^2 + x_2^2 = a_1^2 + a_2^2;$$

$\alpha_1, \alpha_2$  are then the semi-axes of the quadric.

A first generalization of this problem was Hilbert's theory of quadratic forms in infinitely many variables (1906), with application to integral-equations. Hilbert's theory was also formulated in terms of infinite matrices. Infinite matrices were later on basic in the quantum theory of Heisenberg (1925); they were, however, of an essentially more general character than those that could be treated by Hilbert's theory.

Quite a different class of problems in analysis present strong analogies to the problem of principal axes of quadratic forms: namely, the problems concerning characteristic values of differential equations and the connected expansions of the type of Fourier's series and Fourier's integral. A typical problem of this class can be formulated as follows: We take the differential equation  $-\frac{d}{ds}x(s) = \lambda x(s)$ ,

where  $\lambda$  is a value to be determined, and consider two classes of admitted functions.

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Mathematical Analysis

The theory of the differential calculus is a branch of mathematics which deals with the study of the variation of functions of one or more variables.

$$y = f(x) = x^2 + 2x + 1$$

Let  $y = f(x)$  be a function of  $x$ . Then the differential of  $y$  with respect to  $x$  is denoted by  $dy$  and is defined by the equation

$$dy = f'(x) dx$$

$$d^2y = f''(x) dx^2$$

The differential of a function of two variables  $z = f(x, y)$  is denoted by  $dz$  and is defined by the equation

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

The partial differential of a function of three variables  $u = f(x, y, z)$  is denoted by  $du$  and is defined by the equation

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

The total differential of a function of three variables  $u = f(x, y, z)$  is denoted by  $du$  and is defined by the equation

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1. We restrict  $s$  to the range  $0 \leq s \leq 1$  and impose the boundary condition  $x(0) = x(1) = 0$ . Then the solutions are

$$x = u_n(s) = 2^{-1/2} \sin n\pi s, \quad n = 1, 2, 3, \dots$$

$$\lambda = \lambda_n = n^2 \pi^2 \quad \text{"eigen-values"}$$

With them we have the expansions.

$$x(s) = \sum_n a_n u_n(s); \quad \int_0^1 x(s)^2 ds = \sum_n a_n^2$$

$$-\frac{d^2}{ds^2} x(s) = \sum_n a_n u_n(s); \quad \int_0^1 \left(\frac{d}{ds} x(s)\right)^2 ds = \sum_n \lambda_n a_n^2$$

The first expansion is simply a Fourier series. The latter two formulas present an obvious analogy to the reduction of quadratic forms to principal axes. The set of values  $\lambda_n$  is called the spectrum of  $-\frac{d^2}{ds^2}$ ; in particular it is a point spectrum

since it consists of discrete points on the  $\lambda$ -axis.

2. Secondly we restrict  $s$  to the range  $0 \leq s < \infty$  and impose as boundary condition only  $x(0) = 0$ . Then the solutions are

$$x(s) = u_\mu(s) = (\pi/2)^{1/2} \sin_\mu s,$$

where  $\mu$  runs over the entire range  $0 \leq \mu < \infty$ ,  $\lambda = \lambda_\mu = \mu^2$ , and we have the expansions

$$x(s) = \int_0^\infty a_\mu u_\mu(s) d\mu; \quad \int_0^\infty x(s)^2 ds = \int_0^\infty a_\mu^2 d\mu$$

$$-\frac{d^2}{ds^2} x(s) = \int_0^\infty a_\mu u_\mu(s) d\mu; \quad \int_0^\infty \left(\frac{d}{ds} x(s)\right)^2 ds = \int_0^\infty \lambda_\mu a_\mu^2 d\mu.$$

The first expansion is nothing but a Fourier-Integral.

In this case where the characteristic values run over a continuous range  $0 \leq \lambda < \infty$ , the spectrum is termed "continuous". It is interesting to note that here the characteristic functions themselves are not representable because the integral  $\int_0^\infty u_\mu^2(s) d\mu$  is infinite - a point which was left in obscurity for some time.

Let  $f(x) = x^2 + 2x + 1$  and  $g(x) = x^2 - 10x + 16$ .  
 Then  $f(x) + g(x) = (x^2 + 2x + 1) + (x^2 - 10x + 16)$   
 $= 2x^2 - 8x + 17$

$$f(x) + g(x) = 2x^2 - 8x + 17$$

Therefore, the sum of the functions is  $2x^2 - 8x + 17$ .

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The classical application of problems of the preceding type is that to natural vibrations of elastic media. In 1926 Schrödinger caused quite a stir when he presented his quantum theory based on differential equations of a similar though more complicated type. Although Schrödinger himself showed the formal identity of his and Heisenberg's theory, a complete mathematical theory of which all the mentioned theories are special cases, was lacking.

It was the decisive merit of v. Neumann to have broken the mathematical deadlock. He discovered that this deadlock was due to the use of a wrong system of concepts and he developed an appropriate new system of concepts. Using it, he and Stone (1929) derived a general theory of spectral resolution. It is true, von Neumann and Stone did not show that the differential equation problems of quantum mechanics fall under their general theory; to show this has been one of my personal interests. The decisive step, however, was the discovery of the right system of concepts by v. Neumann. Strongly enough, physicists have apparently failed to appreciate von Neumann's work sufficiently. This lack of appreciation may partly be due to the fact that v. Neumann's system is the right one for formulating the principles of quantum theory, not, however, the right one for actually calculating solutions of special problems.

Quadratic forms	Differential equations
Infinite matrices Hilbert	Fourier series
Integral equations 1906	Fourier integral
Quantum theory Heisenberg 1925	Quantum theory Schrödinger 1927
Operators in Hilbert spaces	
v. Neumann 1927/29	
Stone 1929	

In this course I do not want to derive methods for actually calculating solutions of special characteristic value problems. On the other hand I do not want to derive the abstract concepts of von Neumann in their full generality. Rather I wish to derive them gradually in immediate connection with special problems, in particular, differential equation problems; and I want to show how naturally all existence, completeness and expansion theorems result from a general theory.



The following is a list of the names of the persons who have been elected to the office of Justice of the Peace for the year 1900. The names are given in alphabetical order of their surnames. The names of the persons who have been elected to the office of Justice of the Peace for the year 1900 are: [illegible names]

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Literature von Neumann, Mathematische Grundlagen der Quantenmechanik. 1932

Stone, Linear Transformations in Hilbert Space. 1932

### Chapter I. Space of finite dimension.

Space of vectors  $x = \{x_1, x_2, \dots, x_m\}$ ;

length  $|x| = (x, x)^{1/2}$ , where

$x, x = x_1^2 + x_2^2 + \dots$  is the unit-form.

Quadratic form

$xQx = q_{11}x_1^2 + 2q_{12}x_1x_2 + q_{22}x_2^2 + \dots$ ,  $xQx = 1$  quadric.

Eigen-vector or E-vector  $u$ : normal to  $xQx = \text{const.}$  at

$x = u$ , has direction of  $u$ :  $(1/2) \partial xQx / \partial x =$

$2x = \{q_{11}x_1 + q_{12}x_2 + \dots, q_{21}x_1 + q_{22}x_2 + \dots, \dots\}$ ,  $q_{12} = q_{21}$ .

$Q$  operator: transforms vector  $x$  into  $Qx$ ;

$Q$  is linear:  $Q(\alpha x + \beta y) = \alpha Qx + \beta Qy$ .

$Q$  given by matrix  $q_{\sigma\tau}$ .

$u$  is E-vector if

$Qu = Ku$ ;  $K$  Eigen-value or E-value.

Example: Ellipse  $xQx = x_1^2/\alpha_1^2 + x_2^2/\alpha_2^2 = 1$ .

E-vectors  $u^1 = \{1, 0\}$ ,  $u^2 = \{0, 1\}$ ;

E-values  $K_1 = \alpha_1^{-2}$ ,  $K_2 = \alpha_2^{-2}$ .

Circle,  $xQx = (x_1^2 + x_2^2)/R^2 = 1$ . Every vector is

E-vector since  $Qx = \{R^{-2}x_1, R^{-2}x_2\} = R^{-2}x$ .

Relation  $Qx - Kx = 0$  is

$(q_{11} - K)x_1 + q_{12}x_2 + \dots = 0$ ,  $q_{21}x_1 + (q_{22} - K)x_2 + \dots = 0$ .

Condition for solvability is secular equation, (for  $m=2$ ):

$$\begin{vmatrix} q_{11} - K & q_{12} \\ q_{21} & q_{22} - K \end{vmatrix} = 0,$$

quadratic equation (of order  $m$  in general) for  $K$ .

Problem 1. Prove that solutions of secular equation are real.

Example:  $2x_1^2 + 12x_1x_2 - 7x_2^2 = 1$ .

$(2 - K)x_1 + 6x_2 = 0$ ,  $6x_1 - (7 + K)x_2 = 0$ .

$(2 - K)(7 + K) + 36 = 0$ .  $K^2 + 5K - 50 = 0$

$K_1 = 5$ ,  $K_2 = 10$ .  $u^1 = \{2, 1\}$ ,  $u^2 = \{1, -2\}$ .





Preceding procedure best for numerical determination; to be abandoned as theoretical approach since it can be generalized only to restricted class of quadratic forms in spaces of infinite dimension.

Maximum problem. Consider quotient

$xQx/x, x$ ; for  $x \neq 0$ . Unchanged when  $x$  replaced by  $cx$ ; hence  $x, x = 1$  no restriction

Geometric meaning: since no restriction we assume  $xQx = \text{const}$ ; i.e.  $x$  to be on quadric, quotient is reciprocal square of distance between point on quadric and origin. For minimum distance E-vector expected.

Quotient is bounded; assume  $x, x = x_1^2 + x_2^2 + \dots = 1$ ,

then  $|x_n| \leq 1$ .  $xQx \leq |q_{11}| + 2|q_{12}| + \dots$ . Hence:

$xQx/x, x$  has maximum, let its value be  $K$ , attained for  $x = u$ , no restriction:  $|u| = 1$ . Condition:

$$0 = 1/2 \partial [xQx/x, x] / \partial x^{\bar{u}} = (u, u)^{-2} [qu(u, u) - (uqu)u] = qu - Ku. \text{ Hence we have proved.}$$

Theorem 1.1 There exists one E-vector  $\neq 0$ .

Spectral resolution: To give all E-vectors.

Completeness: Manifold of all E-vectors  $q_{11}^2 + q_{22}^2 + \dots = 1$

Orthogonality: E-vectors to different E-values are  $\perp$ .

Both properties to be established.

Inner product:

$$x, y = x_1 y_1 + x_2 y_2 + \dots, \quad x \perp y \text{ means } x, y = 0.$$

Bilinear form:

$$xQy = x_1 q_{11} y_1 + x_1 q_{12} y_2 + x_2 q_{11} y_1 + x_2 q_{22} y_2 + \dots,$$

$$q_{12} = q_{21}, \dots$$

Symmetry of form:  $xQy = yQx$ ; due to  $q_{12} = q_{21}, \dots$

Green's formula:  $x, Qy = (x, Qy) = (xQ, y)$

Symmetry of operator:  $x, Qy = Qx, y$ .

The first part of the paper is devoted to a discussion of the  
 various methods of determining the rate of reaction. It is shown  
 that the most reliable method is the one which involves the  
 measurement of the change in concentration of the reactants  
 or products. This method is applicable to all reactions, but  
 it is often difficult to apply it to reactions which are very  
 fast or very slow. In such cases, other methods must be used.  
 The second part of the paper is devoted to a discussion of the  
 factors which influence the rate of reaction. It is shown that  
 the rate of reaction is affected by the concentration of the  
 reactants, the temperature, the presence of a catalyst, and the  
 surface area of the reactants. The third part of the paper is  
 devoted to a discussion of the mechanism of reaction. It is  
 shown that the mechanism of reaction is the sequence of steps  
 which lead from the reactants to the products. The fourth part  
 of the paper is devoted to a discussion of the kinetics of  
 reaction. It is shown that the kinetics of reaction is the study  
 of the rate of reaction and the factors which influence it.

The rate of reaction is defined as the change in concentration  
 of the reactants or products per unit time. It is usually  
 expressed in terms of moles per liter per second. The rate of  
 reaction is affected by the concentration of the reactants, the  
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Theorem 1.2 E-vectors to different E-values are  $\perp$ .

Proof: Let  $Qu^1 = \kappa_1 u^1$ ,  $Qu^2 = \kappa_2 u^2$ , then

$$u^1, Qu^1 = \kappa_1 (u^1, u^1); u^1, Qu^2 = \kappa_2 (u^1, u^2);$$

difference:  $(\kappa_1 - \kappa_2) (u^2, u^1) = 0$ ; hence  $u^2 \perp u^1$  if  $\kappa_1 \neq \kappa_2$

Lemma 1.1 If  $x \perp$  E-vector  $u$  then  $Qx \perp u$ .

Proof  $u, Qx = uQx = xQu = x, Qu = \kappa (x, u) = 0$ .

Theorem 1.3 E-vectors span whole space.

Proof. If they did not, subspace  $\mathcal{G}^*$  of all  $x^*$  which are  $\perp$  to all E-vectors would have dimension  $\geq 1$ .

Lemma 1.1 shows that  $Qx$  is also in  $\mathcal{G}^*$ . Apply Theorem

1.1 to  $Q$  considered an operator in  $\mathcal{G}^*$ ; it follows that there would be an E-vector  $u \neq 0$  in  $\mathcal{G}^*$ . However,

$\mathcal{G}^*$  cannot contain such E-vectors since it is  $\perp$  to them. Contradiction,

Problem2: Analyze the steps hidden in this reasoning.

Herewith the spectral resolution is completed.

Spectral representation:

Lemma 1.2 Let  $u, v$  be E-vectors to same E-value  $\kappa$  then linear combination  $\alpha u + \beta v$  is E-vector, too.

Proof: Linearity of  $Q$ .

Take all E-vectors to same E-value  $\kappa$ ; they form "E-subspace"  $\mathcal{G}_\kappa$  to  $Q$ ;  $\mathcal{G}_\kappa$  can be spanned by normed orthogonal E-vectors  $u$ .

All normed E-vectors obtained this way are  $\perp$  in view of Th. 1.2; they form a coordinate system:  $u^1, u^2, \dots$ . Let  $\kappa_1, \kappa_2, \dots$  be the corresponding E-values (not necessarily different). Then each vector  $x$  can be written:

$$x = a_1 u^1 + a_2 u^2 + \dots + a_m u^m; \text{ or } x = \{a_1, a_2, \dots\};$$

the operation  $Qx$  may be written, since  $Q$  is linear,

$$Qx = \kappa_1 a_1 u^1 + \kappa_2 a_2 u^2 + \dots \text{ or } Qx = \{\kappa_1 a_1, \kappa_2 a_2, \dots\}.$$

The unit-form and the quadratic form become

$$x, x = a_1^2 + a_2^2 + \dots$$

$$x, Qx = \kappa_1 a_1^2 + \kappa_2 a_2^2 + \dots$$

Thus the spectral representation is completed.

1. (1)  $\mu$  is a  $\sigma$ -finite measure on  $\mathcal{A}$  and  $\nu$  is a  $\sigma$ -finite measure on  $\mathcal{B}$ . (2)  $\mu$  is a  $\sigma$ -finite measure on  $\mathcal{A}$  and  $\nu$  is a  $\sigma$ -finite measure on  $\mathcal{B}$ .

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(iii)  $\mathcal{A} \subseteq \mathcal{B}$  and  $\mathcal{B} \subseteq \mathcal{A}$  implies that

1.  $\frac{1}{2} \log \frac{1}{2} = -0.5$

1. *Microtus pennsylvanicus* (Zittel)

## Chapter II Hilbert space,

Original Hilbert space  $\mathcal{H}_0$ : all vectors with infinitely many components.

$$x = \{x_1, x_2, \dots\}, \quad \text{for which}$$

$$(x, x) = x_1^2 + x_2^2 + \dots < \infty.$$

Space of infinitely many dimensions.

Not suitable for differential operators and quantum theory.

Von Neumann's "abstract" Hilbert space; to be developed.

Linear space  $\mathcal{L}$  of elements or vectors  $x$ .

Multiplication by real numbers and addition defined.

$$\alpha x + \beta y = z \quad \text{in } \mathcal{L}, \quad \text{if } x, y \text{ in } \mathcal{L}; \alpha, \beta \text{ real.}$$

Vector "0". Axioms.

In what follows space is supposed to imply linearity.

Examples. 1. Space of finite dimension.

2. Space of all  $\{x_1, x_2, \dots\}$ , with or without restriction  $(x, x) < \infty$ .

3. Function spaces:  $x$  is function  $x(s)$  of real variable  $s$ , defined in interval  $I$ , say  $-\infty < s < \infty$  or else.

Space of continuous functions  $\mathcal{C}_I$ , of continuously differentiable functions  $\mathcal{D}_I$ . Notation:  $D = d/ds$ .

Form (bilinear)

$$x \underline{B} y$$

Symmetry:  $x \underline{B} y = y \underline{B} x$ .

Linearity:  $x \underline{B} (\beta y + \gamma z) = \beta (x \underline{B} y) + \gamma (x \underline{B} z)$ ;

the same for left vector due to symmetry.

Non-negative form  $\underline{P}$ :

$$x \underline{P} x \geq 0$$

$$\text{Form } 0 \leq (\alpha x + \beta y) \underline{P} (\alpha x + \beta y) = \alpha^2 (x \underline{P} x) + 2\alpha\beta (x \underline{P} y) + \beta^2 (y \underline{P} y)$$

Schwarz inequality (SI):  $|x \underline{P} y|^2 \leq (x \underline{P} x)(y \underline{P} y)$ , and

Triangular inequality (TI):  $|(\alpha x + \beta y) \underline{P} (\alpha x + \beta y)|^{1/2} \leq |\alpha x \underline{P} \alpha x|^{1/2} + |\beta y \underline{P} \beta y|^{1/2}$ .

Form is positive-definite if

$$x \underline{P} x > 0 \quad \text{for } x \neq 0.$$



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Examples: In finite space  $\sum_{\sigma, \tau} a_{\sigma\tau} x_{\sigma} x_{\tau}$  is bilinear form, symmetric if  $a_{\sigma\tau} = a_{\tau\sigma}$ ;  $x, y = x_1 y_1 + \dots + x_m y_m$  is positive-definite form.

In  $\mathcal{H}_0$  we can define inner product by

$$x, y = x_1 y_1 + x_2 y_2 + \dots$$

Problem: Prove absolute convergence of  $(x, y)$  from  $(x, x) < \infty$ ,  $(y, y) < \infty$ .  $(x, y) \leq |x| |y|$

In space of functions  $x(s)$  we may define bilinear form as follows

1.  $\int x(s) r(s) y(s) ds$ , e.g.  $r = 1$ .
2.  $\int \int x(s) k(s, t) y(t) ds dt$ ,  $k(s, t) = k(t, s)$ , e.g.  $k = e^{-|s-t|}$ .
3.  $\int Dx(s) p(s) Dy(s) ds$  e.g.  $p = 1$ .

Metrization Take one positive-definite form and define it as inner product  $x, y$ ; unit-form:

$$x, x > 0 \quad \text{if} \quad x \neq 0,$$

Norm:  $\|x\| = (x, x)^{1/2}$ , distance  $\|x - y\|$ .

$$\left. \begin{array}{l} \text{SI} \quad |x, y| \leq \|x\| \|y\| \\ \text{TI} \quad \|x + y\| \leq \|x\| + \|y\| \end{array} \right\} \text{except if } \alpha x + \beta y = 0.$$

Examples in function spaces

$$x, x = \int r(s) x^2(s) ds, \quad r > 0 \quad \text{used for } \mathcal{L}_2,$$

$$x, x = \int \{p(s) Dx^2(s) ds + q(s) x^2(s)\} ds, \quad p(s) > 0, q(s) > 0, \text{ used for } \mathcal{H}_0. \text{ (may require restriction).}$$

Limit vector  $x$  of sequence  $x^\sigma$ :  $\|x^\sigma - x\| \rightarrow 0$  as  $\sigma \rightarrow \infty$ . Also

$x^\sigma \rightarrow x$ ; convergence (strong); implies

$$\|x^\sigma\| \rightarrow \|x\| \quad (\text{from TI}),$$

$$y, x^\sigma \rightarrow y, x \quad (\text{from SI}).$$

Examples. In  $\mathcal{H}_0$ , convergence of each component not sufficient.

In  $\mathcal{L}_2$  (with above norm)  $\int r [x^\sigma - x]^2 ds \rightarrow 0$  (i.e. in the mean).



Cauchy sequence  $x^\sigma$ ;  $\|x^\sigma - x^\tau\| \rightarrow 0$  as  $\sigma, \tau \rightarrow \infty$ .

If  $x^\sigma \rightarrow x$  then  $x^\sigma$  is Cauchy sequence.

Space is complete if each Cauchy sequence has limit;

Complete space is termed Euclidean or general Hilbert space (with reference to norm of the type  $\|x\| = (x, x)^{1/2}$ ).

$L_2$  is complete, hence Euclidean.

Above function spaces are not complete. Three approaches:

1. Function spaces can be completed by adjoining functions with squares integrable in Lebesgue's sense.

2. Theory can be worked out in incomplete spaces, cf. Courant-Hilbert Vol. II Ch. VII.

3. Completing by adjoining ideal elements.

Consider space  $\mathcal{R}$  with unit-form  $(x, x)$ . Let  $x^\sigma$  be a Cauchy sequence. Either it has limit vector in  $\mathcal{R}$  or else assign to it ideal limit vector  $x$ . If for two sequences  $x^\sigma, x^\tau$ :  $\|x^\sigma - x^\tau\| \rightarrow 0$ , assign the same limit vector to them.

Define  $\alpha x + \beta y = \lim (\alpha x^\sigma + \beta y^\sigma)$ . In this way space  $\mathcal{R}$  can be extended to a space  $\bar{\mathcal{R}}$ .

$(x, y)$  and  $\|x\|$  can be defined in  $\bar{\mathcal{R}}$ :

From II one derives:

$\|x^\sigma\| - \|x^\tau\| \leq \|x^\sigma - x^\tau\| \rightarrow 0$ , hence  $\|x^\sigma\|$  is Cauchy sequence and therefore bounded. From III one derives for two Cauchy sequences

$x^\sigma, y^\sigma$ :  $\|x^\sigma, y^\sigma - x^\tau, y^\tau\| \leq \|x^\sigma - x^\tau\| \|y^\sigma\| + \|x^\tau\| \|y^\sigma - y^\tau\| \rightarrow 0$ , hence  $(x^\sigma, y^\sigma)$  is Cauchy sequence and therefore converges to limit number. Let  $x = \lim x^\sigma$ ,  $y = \lim y^\sigma$ . If  $x$  and  $y$  are in  $\mathcal{R}$ ,  $(x, y) \rightarrow (x, y)$ ; otherwise define  $(x, y) = \lim (x^\sigma, y^\sigma)$ . Prove that  $(x, y)$  is symmetric and bilinear, further  $\|x\| \geq 0$ . Moreover  $\|x\| > 0$  for  $x \neq 0$ .

$\|x\| = 0$  would mean  $\|x^\sigma\| \rightarrow 0$ , and  $x^\sigma$  would have limit vector 0 in  $\mathcal{R}$ .

$\bar{\mathcal{R}}$  is complete. Let  $\bar{x}^\sigma$  be Cauchy sequence in  $\bar{\mathcal{R}}$ ,  $\|\bar{x}^\sigma - \bar{x}^\tau\| \leq \varepsilon(\sigma) \rightarrow 0$  for  $\sigma \geq \tau$ ; choose  $x^\sigma$  in  $\mathcal{R}$  such that  $\|x^\sigma - \bar{x}^\sigma\| \leq \varepsilon(\sigma)$ , then

$\|x^\sigma - x^\tau\| \leq 2\varepsilon(\sigma) + \varepsilon(\tau) \rightarrow 0$ ; Let  $x = \lim x^\sigma$ , then

$\|\bar{x}^\sigma - x\| \leq \|x^\sigma - x\| + \varepsilon(\sigma) \rightarrow 0$ .

Reference: F. Hausdorff Mengenlehre 2<sup>nd</sup> ed. 1927. § 21.3.



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Example The space  $\mathcal{L}_2$  with unit-form  $(x, x) = \int_0^1 x^2 ds$  may be extended to complete space  $\overline{\mathcal{L}_2}$ , the space  $\mathcal{G}_2$  with unit-form  $\int_0^1 \{pDx^2 + qx^2\} ds$  to complete space  $\overline{\mathcal{G}_2}$ . We do not investigate whether the ideal adjointed vector can be realized by functions.

Subspace of  $\mathcal{H}_2$  is closed if it contains all of its limit vectors. Subspace is dense in  $\mathcal{H}_2$  if each vector of space  $\mathcal{H}_2$  is limit vector of subspace.

Examples. In  $\mathcal{H}_2$ : vectors for which  $x_1 = 0$  form closed subspace, vectors which have only finite number of non-vanishing coefficients form dense subspace.

In  $\overline{\mathcal{L}_2}$ : Dense subspaces are:  $\mathcal{L}_2$  and all piecewise linear functions; further: analytic functions, polynomials, (if  $\mathcal{L}$  is bounded).

Problem: Prove that subspace in  $\mathcal{G}_2$  or  $\mathcal{H}_2$  of functions vanishing at a fixed point is dense in  $\overline{\mathcal{L}_2}$ , not dense in  $\overline{\mathcal{G}_2}$  respectively.

Proof for case  $\mathcal{L} = (0 \leq s \leq 1)$ ,  $\|x\|^2 = \int_0^1 x^2 ds$  or  $\int_0^1 \{Dx^2 + x^2\} ds$  respectively; fixed point:  $x = 0$ .

1. Let  $\eta_\tau(s) = 1 - \tau^{-1}s$  for  $0 \leq s \leq \tau$ ,  $= 0$  for  $s > \tau$ .

Let  $\bar{x}$  be any vector in  $\mathcal{L}$ ,  $x$  in  $\mathcal{L}$  such that  $\|x - \bar{x}\| < \varepsilon/2$ .

Set  $x^\tau = (1 - \eta_\tau)x$ ;  $x^\tau(0) = 0$ ;  $\|x - x^\tau\|^2 = \int_0^\tau \eta_\tau^2 x^2 ds \leq \int_0^\tau x^2 ds \leq \frac{\varepsilon}{2}$

if  $\tau = \tau_\varepsilon$  sufficiently small. Then  $\|x_{\tau_\varepsilon} - \bar{x}\| \leq \|x - \bar{x}\| + \|x_{\tau_\varepsilon} - x\| \leq \varepsilon$ .

2. First prove (\*)  $|x(0)| \leq 2\|x\|$  for  $x$  in  $\mathcal{L}$ .

$|x(0)|^2 = [x(s) + \int_0^s Ds(s') ds']^2 \leq 2x^2(s) + 2\int_0^s Ds^2(s') ds$  see TV10

$|x(0)|^2 \leq 2\int_0^1 x^2(s) ds + 2\int_0^1 Ds^2(s) ds = 2\|x\|^2$ .

Now let  $y$  be in  $\mathcal{V}$  with  $y(0) \neq 0$ . If it could be approximated by  $x^\varepsilon$  in  $\mathcal{V}$  with  $x^\varepsilon(0) = 0$ , contradiction  $|y(0)| \leq 2\|x^\varepsilon - y\| \rightarrow 0$  would result from (\*).

Subset spans (determines) space, if linear combinations are dense.

Examples:  $\mathcal{H}_2$  is spanned by coordinate system; i.e. by set of vectors  $\{0, 0, \dots, 0, 1, 0, \dots\}$ .

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$\overline{L}_S$  is spanned by piecewise linear functions with peak values 1 and rational-salient points, or, for bounded  $\int$ , by powers  $1, x, x^2, \dots$ , or, for  $S: 0 \leq x \leq 1$ , by  $\sin n\pi x$ .

Space is denumerable if it is spanned by denumerable set.

(Proper Hilbert space).

Examples:  $L_2$  and  $\overline{L}_S$  are denumerable.

Problem Prove that  $\overline{L}_S$  is denumerable. ~~see above~~

Projection. Let  $\mathcal{Y}$  be closed subspace of vectors  $y$ . Projection  $Px$  of  $x$  in  $\mathcal{Y}$  is vector in  $\mathcal{Y}$  such that

$$x - Px \perp \mathcal{Y}.$$

Minimum Property:

$$\|x - Px\| < \|x - y\| \quad \text{for all } y \neq Px \text{ in } \mathcal{Y}.$$

$$\begin{aligned} \text{Proof: } \|x - y\|^2 &= \|x - Px\|^2 + (x - Px, y - Px) + \|y - Px\|^2 \\ &= \|x - Px\|^2 + \|y - Px\|^2, \end{aligned}$$

because of  $x - Px \perp y - Px$ .

There is only one projection  $Px$ . Otherwise, from  $x - P_1x \perp \mathcal{Y}$ ,  $x - P_2x \perp \mathcal{Y}$ , contradiction  $P_1x - P_2x \perp \mathcal{Y}$  would result.

Projection theorem 2.1 Let  $\mathcal{Y}$  be a closed subspace in general-Hilbert space  $\mathcal{H}$ ,  $x$  a vector in  $\mathcal{H}$ . Then there is a vector  $Px$  in  $\mathcal{Y}$  such that  $x - Px \perp \mathcal{Y}$ .

Proof (Rellich, F. Riesz 1934). Let  $d$  be the g.l.b. of all  $\|x - y\|$  where  $y$  in  $\mathcal{Y}$ . Then for all  $z$  in  $\mathcal{Y}$

$$0 \leq (\beta z + y - x, \beta z + y - x) - d^2 = \beta^2 z^2 + 2\beta(z, y - x) + \|y - x\|^2 - d^2.$$

i.e. this quadratic function is non-negative; hence

$$(1) \quad |z, y - x| \leq \|z\| \{ \|y - x\|^2 - d^2 \}^{1/2}.$$

Apply to  $y^1$  and  $y^2$ , use  $|z, y^1 - y^2| \leq |z, y^1 - x| + |z, y^2 - x|$ ,

set  $z = y^1 - y^2$ ,

$$(2) \quad \|y^1 - y^2\| \leq \{ \|y^1 - x\|^2 - d^2 \}^{1/2} + \{ \|y^2 - x\|^2 - d^2 \}^{1/2}.$$





Now take minimizing sequence  $y^\sigma$ , i.e.  $\|y^\sigma - x\| \rightarrow d$ , (exists!). From (2) we have  $\|y^\sigma - y^\tau\| \rightarrow 0$ , hence  $y^\sigma$  is Cauchy-sequence. Since  $\mathcal{H}$  is complete, there is a limit vector  $\hat{y}$  such that  $y^\sigma \rightarrow \hat{y}$ ; since  $\mathcal{H}$  is closed,  $\hat{y}$  is in  $\mathcal{H}$ . Evidently  $\|\hat{y} - x\| = d$ . From (1) we have  $\|z, \hat{y} - x\| = 0$ , i.e.  $\hat{y} - x \perp \mathcal{H}$ . Hence  $\hat{y} = Px$  is projection.

Complementary space  $\mathcal{H}^*$  to closed subspace  $\mathcal{H}$  consists of all vectors  $\perp \mathcal{H}$ .

Theorem 2.2  $\mathcal{H} \oplus \mathcal{H}^* = \mathcal{H}$ , i.e.  $\mathcal{H}$  and  $\mathcal{H}^*$  span  $\mathcal{H}$ . Or, each vector  $x$  in  $\mathcal{H}$  can be split into sum of vector in  $\mathcal{H}$  and vector in  $\mathcal{H}^*$ .

In fact:  $x = Px + (x - Px)$ .

Problem: Let  $\mathcal{L}$  be subspace in  $\mathcal{V}$  of functions vanishing at fixed point. Construct complement  $\mathcal{L}^*$  to closure  $\overline{\mathcal{L}}$  of  $\mathcal{L}$  in  $\mathcal{V}$ .

Coordinate system:  $u^1, u^2, u^3, \dots, \|u^\sigma\| = 1; u^\sigma \perp u^\tau, \sigma \neq \tau$ .

Complete if it spans space.

Problem. If space is finite or denumerable, a complete coordinate-system exists.

Theorem 2.3: Let  $\{u^\mu\}$  be a complete coordinate system, then for any  $x$  in

$$x = \alpha_1 u^1 + \alpha_2 u^2 + \dots, \text{ where } \alpha_\mu = (x, u^\mu).$$

$$\|x\|^2 = \alpha_1^2 + \alpha_2^2 + \dots; \text{ i.e.}$$

$\mathcal{H}$  is equivalent to  $\mathcal{H}_0$  or finite.

Proof: Take subspace  $\mathcal{U}_m$  spanned by  $u^1, \dots, u^m$ ;  $P_m x$  projection of  $x$  into  $\mathcal{U}_m$ . Let  $P_m x = \beta_1 u^1 + \dots + \beta_m u^m$ ; then, for  $1 \leq \mu \leq m$ ,  $\alpha_\mu = (x, u^\mu) = (P_m x, u^\mu) = \beta_\mu$ ; hence

$$P_m x = \alpha_1 u^1 + \dots + \alpha_m u^m.$$

Since system  $\{u^\mu\}$  is complete there is sequence  $y^n$  in  $\mathcal{U}_m$  such that  $\|y^n - x\| \rightarrow 0$ . Now  $\|P_m x - x\| \leq \|y^n - x\|$ , hence  $\|P_m x - x\| \rightarrow 0$ , whence  $\|P_m x\| \rightarrow \|x\|$ , but  $\|P_m x\|^2 = \alpha_1^2 + \dots + \alpha_m^2$ , Hence  $\|x\|^2 = \alpha_1^2 + \alpha_2^2 + \dots$ .

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Chapter III. Completely continuous forms and joint spectra.

Hilbert space  $\mathcal{H}$  of vectors  $x$ ; unit form  $(x, x)$

Quadratic form  $\underline{Q}x$  resulting from symmetric

Bilinear form  $x \underline{Q} y = y \underline{Q} x$ . Assume

$0 < x \underline{Q} x < k(x, x)$ ; i.e.  $\underline{Q}$  is positive-definite and bounded.

(Positive-definiteness immaterial).

Lemma 3.1 Let  $x^\sigma \rightarrow x$  then  $y \underline{Q} x^\sigma \rightarrow y \underline{Q} x$ ;  $x^\sigma \underline{Q} x^\sigma \rightarrow x \underline{Q} x$ .

Proof 1.  $|y \underline{Q} (x^\sigma - x)| \leq y \underline{Q} y ((x^\sigma - x) \underline{Q} (x^\sigma - x))^{1/2} \leq y \underline{Q} y k \|x^\sigma - x\|^2 \rightarrow 0$ .

2.  $|(x^\sigma \underline{Q} x^\sigma)^{1/2} - (x \underline{Q} x)^{1/2}| \leq ((x^\sigma - x) \underline{Q} (x^\sigma - x))^{1/2} \leq k^{1/2} \|x^\sigma - x\| \rightarrow 0$ .

Definition of E-values  $\kappa$  and E-vectors  $u$  without reference to operators:

(3.1)  $y \underline{Q} u = \kappa(y, u)$  for all  $y$  in  $\mathcal{H}$ .

Theorem 3.1 Two E-vectors to different E-values  $\kappa$  are  $\perp$ .

Proof:  $\kappa_1(u^1, u^2) = u^1 \underline{Q} u^2 = \kappa_2(u^1, u^2)$ .

Theorem 3.2 E-vectors to same E-value  $\kappa$  form closed space  $\mathcal{U}_\kappa$ .

Proof 1.  $\mathcal{U}_\kappa$  is linear, from (3.1). 2.  $\mathcal{U}_\kappa$  is closed: let  $u^\sigma$  be sequence with limit  $v$ ; then  $(y, u^\sigma) \rightarrow (y, v)$  and, from Lemma 3.1,  $y \underline{Q} u^\sigma = y \underline{Q} v$ ; hence  $y \underline{Q} v = \kappa(y, v)$ , i.e.  $v$  in  $\mathcal{U}_\kappa$ .

Lemma 3.2 Let  $u^1, \dots, u^m$  be an orthonormal set of E-functions with E-values  $\kappa_1, \dots, \kappa_m$ ; then for  $x = \alpha_1 u^1 + \dots + \alpha_m u^m$ ,

$$(x, x) = \alpha_1^2 + \dots + \alpha_m^2, \quad x \underline{Q} x = \kappa_1 \alpha_1^2 + \dots + \kappa_m \alpha_m^2.$$

Proof:  $(x, u^\mu) = \alpha_\mu$ ;  $(x, x) = \sum_\mu \alpha_\mu (x, u^\mu) = \sum_\mu \alpha_\mu^2$ ;

$$x \underline{Q} u^\mu = \kappa_\mu \alpha_\mu; \quad x \underline{Q} x = \sum_\mu \alpha_\mu (x \underline{Q} u^\mu) = \sum_\mu \alpha_\mu \kappa_\mu \alpha_\mu.$$

Theorem 3.3 Let  $\mathcal{U}_\omega$  be space spanned by E-vectors to E-values  $\geq \omega$ ; then  $x \underline{Q} x \geq \omega(x, x)$  for  $x$  in  $\mathcal{U}_\omega$ .

Proof: 1. Let  $x = \alpha_1 u^1 + \dots + \alpha_m u^m$ ; in view of Th. 3.2  $\kappa_1, \dots, \kappa_m$  may be considered different; in view of Th. 3.1 the  $u^\mu$  are perpendicular; then the statement follows from Lemma 3.2. 2. For limits of such linear combinations the statement follows from Lemma 3.1.





Desired type of spectral representation: "positive-discrete" representation:-

Complete orthogonal system of normal  $E$ -vectors  $u^1, u^2, \dots$ ;  
 $u^\mu \perp u^\nu, \|u^\mu\| = 1$ .

Decreasing sequence of positive  $E$ -values  $K_1 \geq K_2 \geq K_3 \geq \dots \rightarrow 0$ .  
 Able to represent each vector in :

$x = \alpha_1 u^1 + \alpha_2 u^2 + \dots$ , where  $\alpha_\mu = (u^\mu, x)$ ; further

$(x, x) = \alpha_1^2 + \alpha_2^2 + \dots$ ,  $(x, x) = K_1 \alpha_1^2 + K_2 \alpha_2^2 + \dots$ .

The latter relation follows from  $P_m x = \alpha_1 u^1 + \dots + \alpha_m u^m \rightarrow x$  and  
 $P_m x P_m x = K_1 \alpha_1^2 + \dots + K_m \alpha_m^2$  (cf. Lemma 3.2) in view of Lemma 3.1.

Above-type of spectral representation implies following properties of spectral resolution:

Pure point spectrum:- set of  $E$ -vectors span whole space.

Positive-discrete: point spectrum  $K > 0$  and  $E$ -vectors to  
 $K \geq \omega > 0$  span finite space; i.e.  $\mathcal{M} \geq \omega$  has finite dimension.

Theorem 3.4 If  $E$  has pure positive-discrete point spectrum  
 then "positive-discrete" representation holds.

Proof Since  $E$ -vectors to each  $E$ -value form finite space, they  
 can be spanned by finite coordinate system. All  $E$ -vectors thus  
 obtained can be ordered such that  $K_1 \geq K_2 \geq K_3 \geq \dots \rightarrow 0$ . The re-  
 sulting system is complete; hence representation follows from  
 Th. 2.3.

Hilbert's discovery: Forms with pure positive-discrete point  
 spectrum can be simply characterized.

Property "ny": To each  $\varepsilon > 0$  there are  $r = r(\varepsilon)$  vectors

$y^1, \dots, y^r$  in  $\mathcal{H}_\gamma$  such that for all  $x$  in  $\mathcal{H}_\gamma$

(W) -  $|(x, x) - \sum_{j=1}^r (y^j, x)^2| \leq \varepsilon (x, x)$ .

Theorem 3.5 "Positive-discrete" representation implies "ny".

Proof: Set  $y^j = K_j^{-1/2} u^j$  for  $K_j \geq \varepsilon$ .

THE UNIVERSITY OF CHICAGO, CHICAGO, ILL.,

DECEMBER 15, 1871.

TO THE EDITOR,

I have the honor to acknowledge the receipt of your letter of the 10th inst.

and in reply to inform you that the same has been forwarded to the

proper authorities for their consideration.

I am, Sir, very respectfully,

Yours very truly,

JOHN D. COOPER, President.

THE UNIVERSITY OF CHICAGO, CHICAGO, ILL.

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We shall show that positive-definite form  $\underline{Q}$  enjoying "W"-has positive-discrete point spectrum and, consequently, a "positive-discrete" representation. Before doing so we mention:

Hilbert had given different-condition instead of "W".

Property "CC" of complete continuity: Let  $x^\sigma$  be a sequence with bounded  $\|x^\sigma\|$  such that  $(y, x^\sigma) \rightarrow 0$  for all  $y$  in  $\mathcal{H}_y$  (weak convergence), then  $x^\sigma \underline{Q} x^\sigma \rightarrow 0$ .

Theorem 3.6 - "W" implies "CC". Evident from "W".

Problem: Prove: "CC" implies "W".

Theorem 3.7 If positive-definite form  $\underline{Q}$  enjoys "W" its point spectrum is positive-discrete.

Proof: Let  $\mathcal{U} \supseteq \omega$  (cf. Th.3.3) have dimension  $> r(\varepsilon)$  for  $\omega > \varepsilon$ : then it contains vector  $x$  such that  $x y^\rho = 0$ ,  $\rho = 1, \dots, r$ .

For this  $x$ , (W) gives contradiction to Th.3.3.

Theorem 3.8 - If positive-definite form  $\underline{Q}$  enjoys "W" it possesses E-vector, provided the space  $\mathcal{H}_y$  possesses one vector  $x \neq 0$ .

Proof based on Maximum-problem for  $x \underline{Q} x / x, x$  for  $x \neq 0$ .

Let  $K > 0$  be least upper bound; then  $K - \underline{Q}$  non-negative form. Consider normed maximizing sequence  $x^\sigma$ ,  $\|x^\sigma\| = 1$ ,  $x^\sigma (K - \underline{Q}) x^\sigma = \varepsilon_\sigma^2 \rightarrow 0$ . Set  $x^\sigma - x^\tau = x^{\sigma\tau}$ . TI applied to  $x^{\sigma\tau}$  for  $K - \underline{Q}$  yields

$$x^{\sigma\tau} (K - \underline{Q}) x^{\sigma\tau} \leq (\varepsilon_\sigma + \varepsilon_\tau)^2.$$

Upon adding (W) for  $x^{\sigma\tau}$ ,

$$(*) - (K - \varepsilon)(x^{\sigma\tau}, x^{\sigma\tau}) \leq \sum_{\rho=1}^n (y^\rho \underline{Q} x^{\sigma\tau})^2 + (\varepsilon_\sigma + \varepsilon_\tau)^2.$$

Choose  $\varepsilon < K$ , then  $r$  and  $y^1, \dots, y^r$ ; further subsequence  $x^\sigma$  such that the  $r$  bounded sequences  $(y^\rho \underline{Q} x^\sigma)$  converge, which implies  $(y^\rho \underline{Q} x^{\sigma\tau}) \rightarrow 0$ . For this subsequence, (\*) yields  $\|x^{\sigma\tau}\| \rightarrow 0$ ; i.e. - it is Cauchy sequence and converges to limit vector  $u$  due to completeness of space.  $\|x^\sigma\| = 1$  yields  $\|u\| = 1$ ,  $x^\sigma (K - \underline{Q}) x^\sigma \rightarrow 0$  yields  $u(K - \underline{Q})u = 0$  (Lemma 3.1). Further from 3I,

$$|y(K - \underline{Q})u|^2 \leq |y(K - \underline{Q})y| |u(K - \underline{Q})u| = 0;$$

hence  $y \underline{Q} u = K(y, u)$  for all  $y$  in  $\mathcal{H}_y$ ; i.e.  $u$  is E-vector with E-value  $K$ .





Theorem 3.9 If positive-definite form  $\underline{Q}$  enjoys property "W" its E-vectors are complete.

Proof Consider closed space  $\mathcal{Z}$  spanned by all E-vectors and complementary space  $\mathcal{Z}^*$  of all  $x \perp \mathcal{Z}$ . Since  $\mathcal{Z}$  and  $\mathcal{Z}^*$  span  $\mathcal{H}$ , (cf. Theorem 2.2), we have to prove  $\mathcal{Z}^* = 0$ , i.e. consisting of  $x = 0$  only. First we observe

(\*)  $t \underline{Q}x$  for any  $t$  in  $\mathcal{Z}$ ,  $x$  in  $\mathcal{Z}_*$ .

This relation holds if  $t = u$  is an E-vector because of  $u \underline{Q}x = K(u, x) = 0$ . It therefore holds for every linear combination of E-vectors and in view of Lemma 3.1 also for every  $t$  in  $\mathcal{Z}$ . Hence

(\*)  $y \underline{Q}x = y_* \underline{Q}x$  where  $y$  is any vector in  $\mathcal{H}$ ,  $y_*$  its projection in  $\mathcal{Z}_*$  and  $x$  any vector in  $\mathcal{Z}_*$ .

We now consider  $\underline{Q}$  a positive-definite form in  $\mathcal{Z}_*$ . An E-vector  $u_*$  of  $\underline{Q}$  in  $\mathcal{Z}_*$  is also E-vector of  $\underline{Q}$  in  $\mathcal{Z}$  because  $y_* \underline{Q}u_* = K_*(y_*, u_*)$  for all  $y_*$  in  $\mathcal{Z}_*$  implies  $y \underline{Q}u_* = K_*(y, u_*)$  for all  $y$  in  $\mathcal{H}$ , according to (\*). Since every E-vector of  $\underline{Q}$  in  $\mathcal{H}$  is  $\perp$  to  $\mathcal{Z}_*$ , there is no E-vector of  $\underline{Q}$  in  $\mathcal{Z}_*$ . However, when relation (\*) is applied to  $x = x_*$  in  $\mathcal{Z}_*$ , one may replace  $y \underline{Q}x$  by  $y_* \underline{Q}x$  where  $y_*$  is the projection of  $y$  in  $\mathcal{Z}_*$ ; This is obvious from (\*). Thus one obtains

$$x_* \underline{Q}x_* \leq \sum_{p=1}^{\infty} (y_p^* \underline{Q}x_*)^2 + \varepsilon(x_*, x_*),$$

which expresses that  $\underline{Q}$  in  $\mathcal{Z}_*$  enjoys "W". Therefore, Th. 3.8 would be applicable and would yield the existence of an E-vector in  $\mathcal{Z}_*$  unless  $\mathcal{Z}_*$  contained only the vector "0". The latter must be the case since there is no E-vector in  $\mathcal{Z}_*$ .

[illegible]



Differential form. Open interval  $J$ :  $s^- < |s| < s^+$  (finite or infinite).

space  $\mathcal{J} = \mathcal{J}_J$  of functions  $x(s)$  with continuous derivative vanishing in neighborhood of endpoints of  $J$ . Unit-form

$$(x, x) = \int_J \{p(s) Dx^2(s) + q(s)x^2(s)\} ds,$$

Quadratic form

$$xQx = \int_J r(s)x^2(s) ds$$

where  $p(s) > 0$ ,  $q(s) \geq r(s)$ ,  $r(s) > 0$  and continuous in open  $J$ .

Complete this space  $\mathcal{J}$  to a space  $\hat{\mathcal{J}} = \hat{\mathcal{J}}_J$  by adjoining ideal vectors.  $\hat{\mathcal{J}}$  implies a boundary condition, the generalization of " $x = 0$  at the endpoints of  $J$ ".

Obviously  $Q$  in  $\hat{\mathcal{J}}$  is positive-definite and bounded:

$$(*) \quad 0 < xQx \leq k(x, x), \quad x \neq 0.$$

It can be extended to  $\hat{\mathcal{J}}$ :

Let  $x^\sigma, y^\sigma$  be Cauchy sequences in  $\mathcal{J}$  approaching  $x, y$  in  $\hat{\mathcal{J}}$ . Then  $x^\sigma Q y^\sigma \leq (x^\sigma Q x^\sigma)^{1/2} (y^\sigma Q y^\sigma)^{1/2} \leq k \|x^\sigma\| \|y^\sigma\| \rightarrow 0$ . Therefore  $x^\sigma Q y^\sigma$  converges, and we may define  $xQy = \lim x^\sigma Q y^\sigma$ . It is easily seen that  $xQy$  is a bilinear form in  $\hat{\mathcal{J}}$  with the bound  $k$ .  $Q$  in  $\hat{\mathcal{J}}$  is obviously non-negative; it is, however, not obvious that  $Q$  is positive-definite; i.e. that never  $xQx = 0$  for  $x \neq 0$  in  $\hat{\mathcal{J}}$ . We postpone the proof of this fact to the next chapter.

It may be noted that it is not true that every bounded-form which is positive-definite in a dense subspace is also positive-definite in the complete space.

Counter-example.  $(x, x) = \int_0^1 \{Dx^2 + x^2\} ds + (Dx(1))^2$ ,  $xQx = \int_0^1 x^2 ds$ . In the space  $\mathcal{J}$  of all  $x(s)$  with continuous derivatives  $Dx$  the form  $Q$  is positive-definite; when this space is completed, this no longer holds. To show this we take the sequence

Let  $f(x) = x^2 + 2x + 1$  and  $g(x) = x^2 - 2x + 1$

Find  $(f+g)(x)$  and  $(f-g)(x)$

Also find  $(fg)(x)$  and  $(f/g)(x)$

$$(f+g)(x) = (x^2 + 2x + 1) + (x^2 - 2x + 1)$$

$= 2x^2 + 2$

$$(f-g)(x) = (x^2 + 2x + 1) - (x^2 - 2x + 1)$$

Find  $(fg)(x)$  and  $(f/g)(x)$

$$(fg)(x) = (x^2 + 2x + 1)(x^2 - 2x + 1)$$

$$= (x^2 + 1)^2 - (2x)^2 = x^4 + 2x^2 + 1 - 4x^2 = x^4 - 2x^2 + 1$$

$$(f/g)(x) = \frac{x^2 + 2x + 1}{x^2 - 2x + 1}$$

$$= \frac{(x+1)^2}{(x-1)^2}$$

$$(g/f)(x) = \frac{x^2 - 2x + 1}{x^2 + 2x + 1}$$

$$= \frac{(x-1)^2}{(x+1)^2}$$

Find the domain of  $f(x)$  and  $g(x)$

Since  $f(x)$  and  $g(x)$  are polynomials, their domains are all real numbers.

Find the range of  $f(x)$  and  $g(x)$

For  $f(x) = x^2 + 2x + 1 = (x+1)^2$ , the range is  $[0, \infty)$ .

For  $g(x) = x^2 - 2x + 1 = (x-1)^2$ , the range is  $[0, \infty)$ .

Find the inverse of  $f(x)$  and  $g(x)$

For  $f(x) = (x+1)^2$ , the inverse is  $f^{-1}(x) = \sqrt{x} - 1$  for  $x \geq 0$ .

For  $g(x) = (x-1)^2$ , the inverse is  $g^{-1}(x) = \sqrt{x} + 1$  for  $x \geq 0$ .



$\pi^\varepsilon(s) = 0$  for  $s \leq 1 - \varepsilon$ ,  $= s - 1 + \varepsilon$  for  $1 - \varepsilon \leq |s| \leq 1$ .  
Then  $(\pi^\varepsilon, \pi^\varepsilon) = \varepsilon + \frac{1}{2}\varepsilon^2 \rightarrow 1$ ,  $\pi^\varepsilon Q \pi^\varepsilon = \frac{1}{2}\varepsilon^2 \rightarrow 0$ .

$$(\pi^\varepsilon, \pi^\varepsilon) = (\varepsilon_2 - \varepsilon_1) + \frac{1}{2}(\varepsilon_2 - \varepsilon_1)^2 + \varepsilon_1(\varepsilon_2 - \varepsilon_1)^2 \rightarrow 0,$$

as  $\varepsilon, \varepsilon_2 \rightarrow 0$ . Hence  $\pi$  is a Cauchy sequence; and  $(\pi, \pi) = 1$ ,  $\pi Q \pi = 0$  for the limit vector.

### Examples.

$$1. \underline{x} \underline{L} \underline{x} = \int_0^1 D x^2 ds, \quad \underline{x} \underline{Q} \underline{x} = \int_0^1 x^2 ds; \quad (\underline{x}, \underline{x}) = \underline{x} \underline{L} \underline{x} + \underline{x} \underline{Q} \underline{x}; \quad \text{or, } = \underline{L} + \underline{Q}.$$

E-vectors:  $\sin n\pi s$ , E-values  $K_n = [1 + n^2 \pi^2]^{-1}$ , Fourier expansion.

$$2. \underline{x} \underline{L} \underline{x} = \int_{-1}^1 (1 - s^2) D x^2 ds, \quad \underline{x} \underline{Q} \underline{x} = \int_{-1}^1 x^2 ds; \quad , = \underline{L} + \underline{Q}.$$

E-vectors; Legendre polynomials,  $K_n = [1 + n(n+1)]^{-1}$ .

$$3. \underline{x} \underline{L} \underline{x} = \int_0^1 \{s D x^2 + n^2 s^{-1} x^2\} ds, \quad \underline{x} \underline{Q} \underline{x} = \int_0^1 x^2 ds; \quad , = \underline{L} + \underline{Q}.$$

Bessel functions  $J_m(\xi_{mn})$ ,  $K_n = [1 + \xi_{mn}^2]^{-1}$ , where  $J_m(\xi_{mn}) = 0$ .

$$4. \underline{x} \underline{L} \underline{x} = \int_{-\infty}^{\infty} \left\{ D x^2 + \frac{1}{4} s^2 x^2 \right\} ds, \quad \underline{x} \underline{Q} \underline{x} = \int_{-\infty}^{\infty} x^2 ds; \quad , = \underline{L} + \underline{Q}.$$

$H_n(s) e^{-s^2/4}$ ,  $H_n$  Hermite polynomials,  $K_n = \left[1 + n \frac{1}{2}\right]^{-1}$ .

$$5. \underline{x} \underline{L} \underline{x} = \int_0^{\infty} \left\{ D x^2 + \frac{1}{4} x^2 \right\} s^2 ds, \quad \underline{x} \underline{Q} \underline{x} = \int_0^{\infty} x^2 s ds; \quad , = \underline{L} + \underline{Q}.$$

$L'_n(s) e^{-s/2}$ ,  $L_n$  Laguerre polynomials,  $K_n = [1 + n]^{-1}$ .

$$6. \underline{x} \underline{Q} \underline{x} = \int_0^{\infty} \left\{ a D x^2 + 2bx D x \right\} s^2 ds, \quad \underline{x} \underline{L} \underline{x} = \int_0^{\infty} x^2 s^2 ds; \quad , = \underline{Q} + c \underline{L},$$

where  $a > 0$ ,  $c > b^2/a$ .

Negative discrete spectrum  $K_n = \lambda_n / c + \lambda_n$ ,  $\lambda_n = -n^{-2} b^2/a$ .

$L'_n(2sb/na) e^{-sb/na}$ ; Schrodinger functions for  $a = \hbar^2/2m$ ,  $2b = e^2$

$$7. \underline{x} \underline{L} \underline{x} = \int_{-\infty}^{\infty} D x^2 ds, \quad \underline{x} \underline{Q} \underline{x} = \int_{-\infty}^{\infty} x^2 ds; \quad , = \underline{L} + \underline{Q}.$$

No discrete spectrum.

ما من شيء من هذه الأشياء إلا وله شأن عظيم في حياة الإنسان  
فإنه لا يمكن أن يعيش الإنسان إلا بهذه الأشياء كلها

والله اعلم بالصواب

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We shall derive three criteria such that the above differential form  $Q$  possess property "W" with respect to the above unit-form  $(x, \pi)$ . Before doing so we observe that the coefficient  $p$  can be made 1 by a transformation. Since  $p > 0$  we may introduce

$$t = \int_s ds' / p(s')$$

as new variable; then

$$(x, \pi) = \int_{t_-}^{t_+} \left\{ \left( \frac{dx}{dt} \right)^2 + q_p \pi^2 \right\} dt, \quad x \pi = \int_{t_-}^{t_+} \pi^2 r_p dt;$$

$q_p$  and  $r_p$  take the place of  $q$  and  $r$ . Note that

$$\int q p ds = \int q_p dt, \quad q_p / r_p = q / r \text{ remain unchanged.}$$

The following is a list of the names of the persons who have been  
 admitted to the office of the Secretary of the Board of Education  
 since the last meeting of the Board.

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Criterion 1.  $Q$  enjoys "W" if  $\int_G r ds < \infty$ ,  $\int_G |t| r ds < \infty$ .

Proof: We introduce  $\int_S r(s') ds' = l(s)$ ,  $\int_S |t(s')| r(s') ds' = j(s)$

Let  $\Delta s$  be a subinterval of  $S$  (finite or infinite), then we set

$\int_{\Delta s} r ds = \delta \ell$ ,  $\int_{\Delta s} |t| r ds = \delta j$ . We then proceed to prove first

Poincare's generalized inequality:

$$\int_{\Delta s} r x^2 ds \leq (\delta \ell)^{-1} \left( \int_{\Delta s} r ds \right)^2 + \delta j \int_{\Delta s} p D x^2 ds$$

for  $x$  in  $\mathcal{D}$ . (even  $x$  in  $\mathcal{D}$ )  
if  $x$  is bounded.

Proof: In view of the preceding remarks we may assume  $p = 1$ ,

$t = s$ . Let  $s_1, s_2$  be in  $\Delta s$ ;  $|x(s_1) - x(s_2)|^2 = \left| \int_{s_1}^{s_2} D x ds \right|^2$

$\leq |s_2 - s_1| \int_{s_1}^{s_2} D x^2 ds \leq [ |s_2| + |s_1| ] \int_{\Delta s} D x^2 ds$ . By integrating with

respect to  $s_1$  and  $s_2$  over  $\Delta s$ ,

$$\int_{\Delta s} \int_{\Delta s} r(s_1) r(s_2) |x(s_1) - x(s_2)|^2 ds_1 ds_2 \leq 2 \delta \ell \delta j \int_{\Delta s} D x^2 ds.$$

The left member is  $2 \delta \ell \int_{\Delta s} r x^2 ds - 2 \left( \int_{\Delta s} r ds \right)^2$ . Hence division

by  $2 \delta \ell$  yields the inequality.

Divide  $S$  into  $R$  intervals  $\Delta_s$  such that all  $\delta \ell j \leq \varepsilon$ . By adding

Poincare-inequalities:

$$\int_S r x^2 ds \leq \sum_{s=1}^R (\delta \ell)^{-1} \left( \int_{\Delta_s} r ds \right)^2 + \varepsilon \int_S D x^2 ds.$$

Set  $z_s^2 = (\delta \ell)^{-1/2}$  in  $\Delta_s$ , = 0 outside; then preceding formula

reads

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$$\int_S r x^2 ds \leq \sum_{q=1}^R (\int_S r z^q ds)^2 + \varepsilon \int_S (Dx)^2 ds.$$

The functions  $z^q$  are not in  $\mathcal{J}$ ; approximate them by functions

$$y^q \text{ in } \mathcal{J} \text{ such that } \int_S r (z^q - y^q)^2 ds \leq \varepsilon^2 / 4R^2 \int_S r (z^q)^2 ds.$$

$$\begin{aligned} \text{Then } (\int_S r z^q ds)^2 - (\int_S r y^q ds)^2 &\leq 2 \int_S r (z^q - y^q) ds \int_S r z^q ds \\ &\leq 2 \left( \int_S r (z^q - y^q)^2 ds \right)^{1/2} \left( \int_S r (z^q)^2 ds \right)^{1/2} \int_S r x^2 ds \leq 3R^{-1} \varepsilon \int_S r x^2 ds. \end{aligned}$$

Inserting we obtain

$$(*) \quad \int_S r x^2 ds \leq \sum_{q=1}^R (\int_S r y^q ds)^2 + \varepsilon \int_S \{ D^2 x + r x^2 \} ds \quad \text{or}$$

$$x^2 \leq \sum_{q=1}^R (y^q(x))^2 + \varepsilon(x, x)$$

This inequality is proved for all  $x$  in  $\mathcal{J}$ . In view of Lemma 3.1

it holds for all  $x$  in  $\mathcal{J}$ . Thus we have proved (W) and establish

Criterion 1.

Application: Examples 1.2.3. The forms  $\underline{Q}$  in these examples

may now be seen to enjoy property "W" with respect to the unit-

form  $\underline{L} + \underline{Q}$  and, therefore, possess a positive discrete spectrum.

We note that the formula (\*) derived for the proof of criterion 1 also holds if all integrations are extended over a subinterval of  $\mathcal{J}$ ; the conditions  $\int r ds < \infty$ ,  $\int |t| r ds < \infty$  need only be satisfied for this sub-interval.  $-\infty < s < s_0$

Criterion 2:  $\underline{Q}$  enjoys "W" if  $q/r \rightarrow 0$  as  $s$  approaches the endpoints.





Proof: To  $\varepsilon > 0$  find an inner subinterval  $\delta_\varepsilon$  outside of which

$$r/q \leq \varepsilon. \quad \text{Then } \int_{s-\delta_\varepsilon}^s r x^2 ds \leq \varepsilon \int_{s-\delta_\varepsilon}^s q x^2 dx \leq \varepsilon \int_{s-\delta_\varepsilon}^s \{Dx^2 + qx^2\} ds.$$

Since formula (\*) of criterion 2 holds for the subinterval  $\delta_\varepsilon$ , addition yields (W).

A similar reasoning leads to

Criterion 3: Q enjoys "V" if at each endpoint either  $\int r ds$  and  $\int |t| r ds$  are finite or if there  $q/r \rightarrow \infty$ .

Application of Criterion 1: Example 4.

Application of Criterion 3: Example 5, where  $r = s$ ,  $q = s^2$ ,

hence  $q/r \rightarrow \infty$  as  $s \rightarrow \infty$ ;  $r \neq s^2$ ,  $t = -s^{-1}$ ;  $\int |t| r ds = s$  is finite for  $s \rightarrow 0$ .

For the discussion of example 6 we observe first that for  $x$  in  $\mathcal{Q}$

$$\int_0^\infty -Dx s^2 ds = -\int_0^\infty x^2 s ds < 0; \text{ hence}$$

$$(x, x) \geq -2b \int_0^\infty x^2 s ds. \quad \text{Further we note that a positive } \alpha \text{ can}$$

be found such that

$$(x, x) \geq \alpha \int_0^\infty \left\{ Dx^2 - xDx + \frac{1}{4} x^2 \right\} s^2 ds.$$

Applying formula (W) of Example 5 we obtain

$$-(x, x) \leq \sum_{k=1}^{\infty} \left( \int_0^\infty y^k x ds \right)^2 + \xi(x, x).$$

1. The first part of the paper is devoted to the study of the properties of the function  $f(x)$  defined by the equation  $f(x) = \int_0^x f(t) dt$ . It is shown that  $f(x)$  is a constant function, and its value is determined by the initial condition  $f(0) = 1$ .

2. In the second part, we consider the function  $g(x)$  defined by the equation  $g(x) = \int_0^x g(t) dt + x$ . It is shown that  $g(x)$  is a linear function, and its value is determined by the initial condition  $g(0) = 0$ .

3. The third part of the paper is devoted to the study of the properties of the function  $h(x)$  defined by the equation  $h(x) = \int_0^x h(t) dt + x^2$ . It is shown that  $h(x)$  is a quadratic function, and its value is determined by the initial condition  $h(0) = 0$ .

4. In the fourth part, we consider the function  $k(x)$  defined by the equation  $k(x) = \int_0^x k(t) dt + x^3$ . It is shown that  $k(x)$  is a cubic function, and its value is determined by the initial condition  $k(0) = 0$ .

5. The fifth part of the paper is devoted to the study of the properties of the function  $l(x)$  defined by the equation  $l(x) = \int_0^x l(t) dt + x^4$ . It is shown that  $l(x)$  is a quartic function, and its value is determined by the initial condition  $l(0) = 0$ .

This formula expresses that  $\underline{Q}$  enjoys property "W-" except that  $\int_0^\infty x^2 ds$  appears instead of  $y^2 \underline{Q} x$ . As we shall see in the next chapter this modified "W-" is equivalent to "W-". From this then it can be deduced that the negative part of the spectrum of  $\underline{Q}$  is discrete.

In Example 7  $\underline{Q}$  possesses no discrete spectrum. To show this we prove that property "CC" is not satisfied. To this effect we need only construct a sequence  $x^\sigma(s)$  such that  $x^\sigma, x^\sigma \rightarrow a \neq 0$ ,  $x^\sigma \underline{Q} x^\sigma \rightarrow b \neq 0$ ,  $y \underline{Q} x^\sigma \rightarrow 0$  for all  $y$  in  $\mathcal{J}$ .

We take  $z(s) = (1 - s^2)^2$  and set

$$x^\sigma(s) = \sigma^{-1/2} z(\sigma^{-1}s - z).$$

We then have

$$x^\sigma \underline{Q} x^\sigma = \int_{-\infty}^{\infty} (x^\sigma)^2 ds = \int_{-\infty}^{\infty} z^2 ds = z \underline{Q} z \neq 0.$$

$$x^\sigma, x^\sigma = \int_{-\infty}^{\infty} \left\{ (Dx^\sigma)^2 + (x^\sigma)^2 \right\} ds = \int_{-\infty}^{\infty} \left\{ \sigma^{-2} (z')^2 + z^2 \right\} ds \rightarrow z \underline{Q} z \neq 0.$$

To show that  $y \underline{Q} x^\sigma \rightarrow 0$ , approximate  $y$  by  $\dot{y}$  in  $\mathcal{J}$ ; then

$$|y \underline{Q} x^\sigma| \leq |\dot{y} \underline{Q} x^\sigma| + \|y - \dot{y}\| \|x^\sigma\|.$$

$\|y - \dot{y}\|$  can be made arbitrarily small;  $\|x^\sigma\|$  is bounded and  $\dot{y} \underline{Q} x^\sigma$  equals zero for sufficiently large  $\sigma$ . Hence the sequence  $x^\sigma$  violates (CC).

Example.  $(z, x) = \int_0^1 x^2(s) ds$ ,  $x \underline{Q} x = \int_0^1 \int_0^1 x(s) k(s, t) x(t) ds dt$ ,  $k(s, t) = k(t, s) \geq 0$  and continuous.

Vibrations of string extended over  $S$ . Let  $p$  be the axial tension,  $r$  mass p.u.l.,  $q$  distributed spring constant.  $k^{-1} = \mu^2$ ,  $\mu$  circular-frequency. Discrete spectrum if total mass  $\int r ds$  and its moment  $\int |s| r ds$  finite or spring constant  $q$  becomes infinite.

Problem. Prove that  $\underline{Q}$  possesses property "V".

Example 8. Buckling of a tapered column.  $S: 0 < s < b$ .

$p = 1$ ,  $q = 0$ ,  $r = (EI)^{-1}$ ;  $E$  modulus of elasticity,  $I = I_b x^4 b^{-4}$

moment of inertia of cross section.  $P$ -applied axial force.

$P^{-1}$   $E$ -value. Prove that here form  $\underline{Q}$  does not enjoy "W".

Problem: Investigate whether or not "W" holds for  $S: 0 < s < \infty$ .

$p$  and  $r = q$  behave like  $s^{\alpha_0}$ ,  $s^{\beta_0}$  or  $s^{\alpha_\infty}$ ,  $s^{\beta_\infty}$  at  $s = 0$ ,  $s = \infty$  respectively.



Let  $f(x)$  be a function defined on  $[a, b]$ . Then the definite integral of  $f(x)$  from  $a$  to  $b$  is denoted by  $\int_a^b f(x) dx$ . The value of this integral is the area under the curve  $y = f(x)$  between  $x = a$  and  $x = b$ .

The definite integral has several important properties. For example, if  $f(x)$  is continuous on  $[a, b]$ , then the integral exists and is unique.

$$\int_a^b f(x) dx = F(b) - F(a)$$

where  $F(x)$  is an antiderivative of  $f(x)$ .

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

or

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$\int_a^b f(x) dx = \int_a^b f(t) dt$$

where  $t$  is a dummy variable.

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k^*) \Delta x$$

where  $\Delta x = (b-a)/n$  and  $x_k^* \in [x_{k-1}, x_k]$ .

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k^*) \Delta x$$

where  $\Delta x = (b-a)/n$  and  $x_k^* \in [x_{k-1}, x_k]$ .

The definite integral is used to find the area under a curve, the volume of a solid, and the work done by a force.

For example, the area under the curve  $y = x^2$  from  $x = 0$  to  $x = 1$  is

$$\int_0^1 x^2 dx = \frac{1}{3} x^3 \Big|_0^1 = \frac{1}{3}$$

which is the area of the region bounded by the curve  $y = x^2$  and the  $x$ -axis from  $x = 0$  to  $x = 1$ .

The definite integral is also used to find the volume of a solid.

For example, the volume of the solid generated by revolving the curve  $y = x^2$  about the  $y$ -axis from  $x = 0$  to  $x = 1$  is



# Chapter IV. Operators.

Hilbert-space  $\mathcal{H}$  of vectors  $x$ . Bounded linear operator assigns to every  $x$  a vector  $Qx$  such that

$$Q(\alpha x + \beta y) = \alpha Qx + \beta Qy$$

and that there is a number  $K$  such that

$$\|Qx\| \leq K\|x\| \text{ for all } x.$$

$Q$  is symmetric if  $(x, Qy) = (Qx, y)$ .

Bilinear form  $Q$  of bounded linear operator  $Q$  defined by

$$xQy = (x, Qy) = (Qx, y); \text{ it is bounded}$$

$$|xQy| \leq K\|x\|\|y\|.$$

Theorem 4.1 To every bounded bilinear form  $Q$  there is exactly one bounded linear operator  $Q$  such that

- (1)  $xQy = (x, Qy)$  holds and
- (2)  $xQy = (x, z)$  implies  $z = Qy$ .

The second statement follows from the first one by setting  $x = z - Qy$ . The proof is based on a lemma concerning bounded linear forms, i.e. forms  $\underline{E}$  with properties

$$\underline{E}(\alpha x + \beta y) = \alpha \underline{E}x + \beta \underline{E}y \text{ and there is a number } \eta \text{ such that}$$

$$|\underline{E}x| \leq \eta\|x\| \text{ for all } x.$$

Lemma 4.1 To every bounded linear form  $\underline{E}$  there is exactly one vector  $e$  in  $\mathcal{H}$  such that

(\*)  $\underline{E}x = (e, x)$  for all  $x$ ; i.e. every linear form can be represented as an inner product with a fixed vector.

The proof is based on the projection theorem: 1. The space  $\mathcal{N}$  of all  $t$  for which  $\underline{E}t = 0$  is closed, because  $t \rightarrow z$  implies

$$|\underline{E}(t^\epsilon - z)| \leq \eta\|t^\epsilon - z\| \rightarrow 0, \quad \underline{E}z = \underline{E}(z - t^\epsilon) \rightarrow 0, \quad \underline{E}z = 0.$$

1. The first part of the document is a letter from the Secretary of the State to the President, dated 18th March 1861. It contains a report on the state of the Union and the progress of the war. The letter is signed by the Secretary and is addressed to the President.

2. The second part of the document is a report from the Secretary of the State to the President, dated 18th March 1861. It contains a report on the state of the Union and the progress of the war. The report is signed by the Secretary and is addressed to the President.

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2. Consider the complementary space  $\mathcal{F}^*$ . If  $\mathcal{F}^* = \emptyset$  then  $\mathcal{F} = \mathcal{H}$  and  $e=0$  satisfies (\*). If  $\mathcal{F}^* \neq \emptyset$  it is spanned by just one vector  $t^*$ : If there were two independent vectors  $t_1^*, t_2^*$  in  $\mathcal{F}^*$ , a linear combination  $e = \alpha_1 t_1^* + \alpha_2 t_2^*$  could be found such that  $\underline{E}t = \alpha_1 \underline{E}t_1^* + \alpha_2 \underline{E}t_2^* = 0$ ; i.e. which would be in  $\mathcal{F}$ . Set  $e = \alpha t^*$  where  $\alpha$  is so chosen that  $\underline{E}e = (e, e)$ ; namely  $\alpha = \underline{E}t^* / (t^*, t^*)$ . Then the projection of any  $x$  into  $\mathcal{F}^*$  is  $(e, e)^{-1} (e, x) e$  and  $x = (e, e)^{-1} (e, x) e + t$  where  $t$  is in  $\mathcal{F}$ . Hence  $\underline{E}x = (e, e)^{-1} (e, x) \underline{E}e = (e, x) e$ .

To prove theorem 4.1 we observe that the bounded form  $xQy$  is a bounded linear form in  $y$  when  $x$  is considered fixed. Hence there is exactly one vector  $z$  such that  $xQy = (z, y)$ ;  $z$  depends on  $x$ ; we set  $z = Qx$ . We have  $(\alpha_1 x^1 + \alpha_2 x^2) Qy = (\alpha_1 Qx^1 + \alpha_2 Qx^2), y$ ; since  $Qx$  is uniquely determined we have  $Q(\alpha_1 x^1 + \alpha_2 x^2) = \alpha_1 Qx^1 + \alpha_2 Qx^2$ ; i.e.  $Q$  is linear. Further we have  $|Qx, y| \leq K \|x\| \|y\|$ ; hence  $\|Qx\|^2 = |Qx, Qx| \leq K \|x\| \|Qx\|$  or  $\|Qx\| \leq K \|x\|$ ; i.e.  $Q$  is bounded. Thus Theorem 4.1 is proved.

Let  $f(x) = x^2 + 2x + 1$  and  $g(x) = x^2 + 1$ . Then

$f(x) - g(x) = (x^2 + 2x + 1) - (x^2 + 1) = 2x$ .

Thus,  $f(x) - g(x) = 2x$ .

Therefore,  $f(x) = g(x) + 2x$ .

Since  $g(x) = x^2 + 1$ , we have

$f(x) = x^2 + 1 + 2x = x^2 + 2x + 1$ .

Thus,  $f(x) = x^2 + 2x + 1$ .

Therefore,  $f(x) = x^2 + 2x + 1$ .

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Therefore,  $f(x) = x^2 + 2x + 1$ .



As a consequence of Theorem 4.1 we remark that property "V" ensues from "W" because  $y \int Qx$  can now be written  $(Qy \int, x)$ . That "W" follows from "V" is not so immediate; these properties are, however, equivalent since both are equivalent to the purely discrete spectral representation.

As an important consequence of Theorem 4.1 the definition of E-vector and E-value  $zQu = K(z, u)$  can be replaced by

$$Qu = \lambda u,$$

as is seen from the second statement in Theorem 4.1 i.e. E-vectors and E-values can be characterized by an operator equation.

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To reveal the nature of the operator  $Q$  we take example 1, for the interval  $a \leq s \leq b$ ; we take, however, the form  $\underline{L}$  for unit-form. That  $Q$  remains bounded in  $\mathcal{J}$  is easily seen.  $(x, x) = \int_a^b D x^2 ds$ ,  $x Q x = \int_a^b x^2 ds$ . We show that  $Qx = \int_a^b k(s, r) x(t) dt$  where the Green's function  $k(s, t) = k(t, s)$  is given by

$$k(s, t) = \begin{cases} (b-a)^{-1}(b-s)(t-a) & \text{for } t \leq s \\ (b-a)^{-1}(s-a)(b-t) & \text{for } t \geq s. \end{cases}$$

Proof: We first prove it for  $x$  in  $\mathcal{J}$ . We have

$$Qx = (b-a)^{-1}(b-s) \int_a^s (t-a)x(t)dt + (b-a)^{-1}(s-a) \int_s^b (b-t)x(t)dt$$

$D^2 Qx = -x$ . For  $y$  in  $\mathcal{J}$  we deduce

$$\int_a^b Dy DQx dx = - \int_a^b y D^2 Qx ds = \int_a^b y x ds \quad \text{or} \quad y \underline{L} Qx = y \underline{L} x.$$

This relation can immediately be extended to  $x$  and  $y$  in  $\mathcal{J}$ .

Let  $f(x) = x^2 + 1$  and  $g(x) = x^2 - 1$ .

Then  $f(x)g(x) = (x^2 + 1)(x^2 - 1) = x^4 - 1$ .

Let  $h(x) = x^4 - 1$ . Then  $h(x) = f(x)g(x)$ .

Let  $h(x) = x^4 - 1$ . Then  $h(x) = f(x)g(x)$ .

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Let  $h(x) = x^4 - 1$ . Then  $h(x) = f(x)g(x)$ .

$$\begin{aligned} \frac{f(x)g(x)}{f(x)} &= \frac{(x^2 + 1)(x^2 - 1)}{x^2 + 1} = x^2 - 1 \\ \frac{f(x)g(x)}{g(x)} &= \frac{(x^2 + 1)(x^2 - 1)}{x^2 - 1} = x^2 + 1 \end{aligned}$$

Let  $h(x) = x^4 - 1$ . Then  $h(x) = f(x)g(x)$ .

$$\frac{f(x)g(x)}{f(x)} = \frac{(x^2 + 1)(x^2 - 1)}{x^2 + 1} = x^2 - 1$$

Let  $h(x) = x^4 - 1$ . Then  $h(x) = f(x)g(x)$ .

$$\frac{f(x)g(x)}{g(x)} = \frac{(x^2 + 1)(x^2 - 1)}{x^2 - 1} = x^2 + 1$$

Let  $h(x) = x^4 - 1$ . Then  $h(x) = f(x)g(x)$ .

Let  $h(x) = x^4 - 1$ .



Inversion of the operator Q. The inverse of the operator  $Q$  in  $\mathcal{H}_y$  can be defined as follows. Let  $Q\mathcal{H}_y$  be the subspace of all vectors  $x$  in  $\mathcal{H}_y$  which are of the form  $x = Qy$  where  $y$  is in  $\mathcal{H}_y$ . The vector  $y$  is uniquely determined; for  $Qy = 0$  implies  $y = 0$  since  $Q$  is positive definite. The operator assigning  $y$  to  $x$  is denoted by  $M$ :  $y = Mx$ . This operator is linear since  $Q(\alpha_1 y^1 + \alpha_2 y^2) = \alpha_1 x^1 + \alpha_2 x^2$  implies  $M(\alpha_1 x^1 + \alpha_2 x^2) = \alpha_1 Mx^1 + \alpha_2 Mx^2$ . Statements (1) and (2) of theorem 4.1 can be written

- (1)  $zQMx = z, x$  holds for  $x$  in  $Q\mathcal{H}_y, z$  in  $\mathcal{H}_y$   
 (2)  $zQy = z, x$  for all  $z$  in  $\mathcal{H}_y$  implies that  $x$  is in  $Q\mathcal{H}_y$  and  $Mx = y$ .

When we apply the latter statement to  $E$ -vectors  $u$ , we see that the relation  $Qu = Ku$  is equivalent to

$u$  is in  $Q\mathcal{H}_y$  and

$Mu = \mu u$  where  $\mu = K^{-1}$  is an " $E$ -value of the operator  $M$ ".

We turn now to the case that  $\mathcal{H}_y$  is the function space  $\mathcal{V}$  and  $Q$  the operator connected with the differential form  $Q$ . We shall see that the inverse  $M$  is a differential operator. For the space  $Q\mathcal{H}_y$  we then use the notation  $\hat{\mathcal{F}}$ .

In what follows we assume that  $r, q, p, \frac{dp}{ds}$  have continuous derivatives. We introduce the space  $\mathcal{F}$  of all functions  $x(s)$  with continuous derivatives up to the third order and the subspace  $\hat{\mathcal{F}}$  of those functions in  $\mathcal{F}$  that vanish identically in a neighborhood of the end points. In  $\mathcal{F}$  we define the differential operator

$M = -r^{-1}[DpD - q]$ , where the coincidence with the inverse of  $Q$  is anticipated in the notation. Then Green's formula

$$zQMx = z, x \quad \text{for } x \text{ in } \hat{\mathcal{F}}, z \text{ in } \hat{\mathcal{V}}$$

holds. For  $z$  in  $\hat{\mathcal{V}}$  this reduces to

$$\int_{\mathcal{S}} z[-DpDx + qx]ds = \int_{\mathcal{S}} \{pDz \cdot Dx + qzx\}ds$$

and is proved through integration by parts since the boundary terms vanish; it then extends to  $z$  in  $\mathcal{V}$ . (Lemma 3.1.)

The space  $\hat{\mathcal{F}}$  is dense in  $\mathcal{V}$ ; this follows from the fact that every function  $x(s)$  in  $\mathcal{V}$  and its derivative can be uniformly approximated by functions in  $\hat{\mathcal{F}}$ ; (prove!).

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We are now in a position to prove the  
 Theorem 4.2 The form  $Q$  is positive definite,  
 which was used in Ch. III. Let  $y$  be a vector in  $\mathcal{D}$   
 with  $yQy=0$ ; then the above Green's formula gives  
 $(y,x)=0$  for all  $x$  in  $\mathcal{F}$ ; since  $\mathcal{F}$  is dense in  $\mathcal{D}$  this holds  
 for all  $x$  in  $\mathcal{D}$ ; in particular for  $x=y$ ; i.e.  $(y,y)=0$  or  
 $y=0$ . This proves Theorem 4.2.

From Theorem 4.1 (2) we see that  $zQMx=z, x$   
 for all  $z$  in  $\mathcal{D}$  implies  $x = QMx$  or  $QM=1$  for  $x$  in  $\mathcal{F}$ .  
 This shows that every function  $x$  in  $\mathcal{F}$  is of the form  
 $x=Qy$ . Therefore  $\mathcal{F}$  is a subspace of  $\mathcal{F} = Q\mathcal{D}$   
 and the differential operator  $M$  in  $\mathcal{F}$  coincides with  
 the inverse of  $Q$ .

The nature of the space  $\hat{\mathcal{F}}$  is revealed by the  
 Theorem 4.31. If  $x$  in  $\mathcal{D}$  then  $x$  can be represented by func-  
 tion in  $\mathcal{L}$ .

4.32. If  $x$  in  $\hat{\mathcal{F}}$  then  $x$  in  $\mathcal{D}$ ,  $Dx$  in  $\mathcal{D}$ ,  $Mx$  in  $\mathcal{L}$ .

4.33 If  $x$  in  $\mathcal{F}$ ,  $Mx$  in  $\mathcal{F}$  then  $x$ ,  $Dx$ ,  $Mx$ ,  $DMx$  in  $\mathcal{D}$ ,  
 $M^2x$  in  $\mathcal{L}$ .

which could be continued.

An  $\mathcal{E}$ -vector  $u$  admits application of  $M$  indefinitely  
 since  $Mu=\mu u$ ,  $M^2u=\mu^2u$ , .... Hence  $u$  is a function in  $\mathcal{D}$   
 with derivative  $Du$  in  $\mathcal{D}$  while the second derivative is in  $\mathcal{D}$   
 due to  $Mu=\mu u$ . This implies the important result that  
 $\mathcal{E}$ -vectors are functions in the original sense and not  
 merely ideal vectors.

For the proof of Theorem 4.31 we may assume  $\mu=1$   
 (as in Ch. III). We first observe that continuous functions  
 $z$  which vanish in the neighborhood of the end points and  
 have piecewise continuous derivatives, belong to  $\mathcal{D}$  because  
 they can be approximated by functions in  $\mathcal{D}$ .

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Let  $S'$  be an inner subinterval of  $S$ , let  $s'$  represent points in  $S'$ . Let  $h(s)$  be a function with continuous derivatives of any order which equals 1 in a neighborhood of  $S'$  and equals zero in a neighborhood of the endpoints. Then

$$k(s) = k(s, s') = \frac{1}{2} h(s) |s - s'|$$

is a function in  $\mathcal{J}$  of the above mentioned kind. The function

$$j = j(s, s') = -r^{-1}(s) \frac{d^2}{ds^2} k(s, s')$$

has then continuous derivatives with respect to  $s'$  of any order. For  $x$  in  $\mathcal{J}$  we obtain

$$(k, x) = \int_S \{Dk Dx + qkx\} ds = \int_S \{j + r^{-1}qk\} x r ds + x(s'), \text{ or}$$

$$A: \quad x(s') = (k, x) - (j + r^{-1}qk) \underline{Q}x, \quad \text{whence}$$

$$|A|: \quad |x(s')| \leq \alpha \|x\|, \quad \text{where } \alpha \text{ depends on } S'.$$

Let now  $x$  in  $\mathcal{J}$  be defined by a Cauchy sequence  $x^\sigma$  in  $\mathcal{J}$ . The inequality  $|A|$  applied to  $x^\sigma - x^\tau$  shows that  $x^\sigma$  converges, uniformly in every subinterval  $S'$ , and hence possesses a continuous limit function  $x(s)$ . This limit function represents the vector  $x$  because 1. it is evidently independent of the defining Cauchy sequence, 2. no two vectors  $x$  possess the same function  $x(s)$ . The latter statement is equivalent to this: no vector  $x \neq 0$  has the limit function  $x(s) \equiv 0$ . This follows from Theorem 4.2 and the identity

$x \underline{Q} x = \int_S r(s) x^2(s) ds$  for the limit function  $x(s)$  of  $x$ . To prove this identity, choose to given  $\varepsilon > 0$  a  $\sigma$  such that

$x^\sigma \underline{Q} x^\tau \leq \varepsilon^2$  for  $\tau > \sigma$ . From this  $\int_{S'} (x^\sigma)^\tau)^2 r ds \leq \varepsilon^2$  whence

$$\int_{S'} (x^\sigma - x)^2 r ds \leq \varepsilon^2 \text{ or } \left| \left( \int_{S'} r x^2 ds \right)^{1/2} - \left( \int_{S'} r (x^\sigma)^2 ds \right)^{1/2} \right| \leq \varepsilon.$$

This holds for every subinterval  $S'$ ; hence

$$\left| \left( \int_S r x^2 ds \right)^{1/2} - (x^\sigma \underline{Q} x^\sigma)^{1/2} \right| \leq \varepsilon. \text{ On the other hand}$$

$$\left| (x \underline{Q} x)^{1/2} - (x^\sigma \underline{Q} x^\sigma)^{1/2} \right| \leq \varepsilon; \text{ hence } \left| \left( \int_S r x^2 ds \right)^{1/2} - (x \underline{Q} x)^{1/2} \right| \leq 2\varepsilon.$$

This proves the identity.



We further observe that formula A and inequality |A| remain valid for  $x$  in  $\mathcal{D}$ .

In order to prove 4.32 we first assume that  $y$  is in  $\mathcal{D}$ .  
 ~~$x=y$~~  We derive from A the formula

$$\begin{aligned} x(s^1) &= \int k y r ds - \int [j + r^{-1} q k] x r ds \\ &= \int k [y - r^{-1} q x] r ds - \int j x r ds. \end{aligned}$$

This formula can be differentiated with respect to  $s^1$  and gives

$$B: D x(s^1) = \frac{1}{2} \int s^1 \eta(s) \operatorname{sgn}(s^1 - s) [y - r^{-1} q x] r ds - \int \frac{dj}{ds^1} x r ds.$$

$$\eta(s) = 0$$

outside an interval

from which

$$|B|: |D(s^1)| \leq \beta' \{ \|y\| + \|x\| \}, \text{ where } \beta' \text{ depends on } s^1.$$

(Observe that  $\left| \frac{dj}{ds^1} \right| = \frac{1}{2} \left| \frac{d^2 \eta}{ds^2} \right|$  is independent of  $s^1$ ).

Now let  $x$  be in  $\mathcal{F}$ ; approximate  $y = Mx$  by functions  $y^\tau$  in  $\mathcal{D}$ , then  $x^\tau = Q y^\tau$  approximates  $x = Q y$ . Upon applying |B| and |A| to  $y^\tau$  and  $x^\tau$  we have

$$\begin{aligned} |D x^\tau(s^1)| &\leq \beta' \{ \|y^\tau\| + \|x^\tau\| \} \rightarrow 0. \\ |x^\tau(s^1)| &\leq \alpha' \|y^\tau\| \rightarrow 0. \end{aligned}$$

Hence the sequence  $x^\tau(s)$  possesses a differentiable limit function  $x(s)$  in every subinterval and, therefore, in  $\mathcal{S}$ . That is, the function  $x(s)$  representing  $x$  in  $\mathcal{F}$  is in  $\mathcal{D}$ . Formula B, which was derived under the assumption  $Mx$  in  $\mathcal{D}$  then will hold also for  $x$  in  $\mathcal{F}$ . This formula can be differentiated another time with respect to  $s^1$  and gives

$$C: D^2 x(s^1) = [ry - qx]_{s=s^1}, \text{ since } \frac{d^2 j}{(ds^1)^2} = 0; \text{ hence}$$

$r^{-1} [-D^2 + q] x = y = Mx$ ; i.e.  $x$  and  $Dx$  are in  $\mathcal{D}$  and  $M$  is the differential operator. That proves Theorem 4.32.



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1. The first of these is the fact that the

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$$\text{and } \frac{\partial}{\partial t} \left( \frac{1}{r^2} \right) = -\frac{2}{r^3} \frac{\partial r}{\partial t} = -\frac{2}{r^3} v_r$$
$$x_{i+1} = \arg \min_{x \in \mathbb{R}^n} \{ \langle \nabla f(x_i), x \rangle + \frac{1}{2} \|x - x_i\|_H^2 \} \quad \forall i \geq 0 \quad (1)$$
$$P_{\text{eff}} = \frac{1}{2} \left( \frac{1}{\pi} \int_0^{2\pi} \frac{1}{1 + \epsilon \cos \theta} d\theta \right) \left( \frac{1}{\pi} \int_0^{2\pi} \frac{1}{1 + \epsilon \cos \theta} d\theta \right) \left( \frac{1}{\pi} \int_0^{2\pi} \frac{1}{1 + \epsilon \cos \theta} d\theta \right)$$

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Let  $\{x_i\}_{i=1}^n$  be a sequence of points in  $\mathbb{R}^d$  such that  $\|x_i - x_j\| \geq \delta$  for all  $i \neq j$ . Then the number of points in the sequence is at most  $\frac{V}{\delta^d}$ , where  $V$  is the volume of the region containing the points.

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$$\Delta_{\text{eff}} = \frac{\Delta}{1 + \left( \frac{\omega_c}{\omega_p} \right)^2} \quad \text{with } \omega_p^2 = \frac{4\pi n e^2}{m}$$

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Supplement to Chapter IV p5 and p,5

Assume that form  $Q$  possess a positive discrete spectrum.

Expansion, with respect to  $E$ -vectors  $u^n$  to  $E$ -values  $K_n$ ,

$$x = \sum_n \alpha_n u^n, \quad \alpha_n = (u^n, x),$$

implies

$$Qx = \sum_n K_n \alpha_n u^n.$$

Proof. Let  $x_m = \sum_{n=1}^m \alpha_n u^n$ , then  $Qx_m = \sum_{n=1}^m K_n \alpha_n u^n$

in view of  $Qu^n = K_n u^n$ . Since  $\|Qx_m - Qx\| \leq K \|x_m - x\| \rightarrow 0$

We have  $\sum_{n=1}^m K_n \alpha_n u^n \rightarrow Qx$ .

For  $x$  in  $Q\mathcal{L}$  we have

$$Mx = \sum_n \mu_n \alpha_n u^n, \quad \mu_n = K_n^{-1}.$$

Proof. Let  $x = Qy$ ,  $y = \sum_n \beta_n u^n$ ,  $\beta_n = (u^n, y)$ ;

then, from preceding,  $x = Qy = \sum_n K_n \beta_n u^n$ .

Now  $K_n \beta_n = K_n (u^n, y) = (Qu^n, y) = (u^n, Qy) = (u^n, x) = \alpha_n$ .

Hence  $\beta_n = \mu_n \alpha_n$ .

Discussion of differential operator  $M$  for examples.

Example 1.  $0 \leq s \leq 1$ ,  $p=1, q=1, r=1$ .

$$M = -D^2 + 1,$$

$E$ -equation  $D^2 u + (\mu - 1)u = 0$  has solutions

$u = c \sin(\tau s + \varphi)$ , where  $\mu = 1 + \tau^2$ .

For which values of  $\tau$  and  $\varphi$  will  $u$  be in  $\mathcal{D}$ ?

1. The first part of the paper is devoted to a general  
discussion of the problem of the existence of solutions  
of the system of equations

$$\frac{dx}{dt} = P(x, y, z), \quad \frac{dy}{dt} = Q(x, y, z), \quad \frac{dz}{dt} = R(x, y, z),$$

where  $P, Q, R$  are functions of  $x, y, z$  which are continuous  
and have continuous first partial derivatives in a certain  
region of space.

It is shown that if the functions  $P, Q, R$  satisfy the  
conditions

$$P^2 + Q^2 + R^2 > 0,$$

then the system of equations has a unique solution

passing through any point of the region.

The second part of the paper is devoted to a study of the

stability of the solutions of the system of equations.

It is shown that if the functions  $P, Q, R$  satisfy the

conditions

$$\frac{dV}{dt} < 0,$$

then the solutions of the system of equations are stable.

The third part of the paper is devoted to a study of the

asymptotic behavior of the solutions of the system of equations.

It is shown that if the functions  $P, Q, R$  satisfy the



Lemma 1. If  $x(s)$  in  $\mathcal{J}$  is in  $\hat{\mathcal{J}}$  then  $x(0) = x(1) = 0$ .

Proof. For  $x(s)$  in  $\mathcal{J}$  we have  $|x(s)|^2 = |x(s_2) + \int_{s_2}^s Dx ds|^2$ ,

$$(*) \quad |x(s_2)|^2 \leq 2|x(s_2)|^2 + 2|s_2 - s_1| \int_{s_1}^{s_2} Dx^2 ds.$$

Set  $s_1 = 0$ ,  $|s_2 - s_1| \leq 1$  and integrate with respect to  $s_2$ .

$$|x(0)|^2 \leq 2 \int_0^1 x^2 ds + 2 \int_0^1 Dx^2 ds \quad \text{or} \quad |x(0)| \leq \|x\|,$$

in the same way  $|x(1)| \leq \|x\|$ . Application to  $x - \dot{x}$ , where  $x$  in

$\hat{\mathcal{J}}$ , gives  $|x(0)| \leq \|x - \dot{x}\|$ ,  $|x(1)| \leq \|x - \dot{x}\|$ .

Since  $\|x - \dot{x}\|$  can be made arbitrarily small, lemma 1 is proved.

Lemma 2. If  $x(0) = x(1) = 0$  for  $x$  in  $\mathcal{J}$  then  $x$  in  $\hat{\mathcal{J}}$ .

Proof. From (\*) in proof of lemma 1 for  $s_1 = \xi$ ,  $s_2 = 0$  and

$$s_1 = 1 - \xi, \quad s_2 = 1 \quad \text{we have} \quad \{x^2(\xi) + x^2(1 - \xi)\} \leq 2\xi \int_0^\xi Dx^2 ds + 2(1 - \xi) \int_{1-\xi}^1 Dx^2 ds.$$

Now replace  $x(s)$  by a function  $x_\xi(s)$  in  $\mathcal{J}$  obtained from

rounding off the function:

$$\begin{aligned} x_\xi(s) &= x(s) && \text{for } \xi \leq s \leq 1 - \xi \\ &= (2\xi^{-1}s - 1)x(\xi) && \text{for } \xi/2 \leq s \leq \xi \\ &= (2\xi^{-1}(1-s) - 1)x(1 - \xi) && \text{for } 1 - \xi \leq s \leq 1 - \xi/2 \\ &= 0 && \text{for } 0 \leq s \leq \xi/2, 1 - \xi/2 \leq s \leq 1. \end{aligned}$$

$$\begin{aligned} \text{Then } \|x - x_\xi\|^2 &\leq 2 \int_0^\xi + \int_{1-\xi}^1 \{Dx^2 + x^2\} ds + 2\xi^{-1}(1 + \xi^2/6) \{x^2(\xi) + x^2(1 - \xi)\} \\ &\leq (6 + \xi^2/3) \int_0^\xi + \int_{1-\xi}^1 \{Dx^2 + x^2\} ds \longrightarrow 0 \text{ as } \xi \longrightarrow 0. \end{aligned}$$

Lemmas 1 and 2 show that those and only those functions

$u = c \sin(\tau s + \phi)$  are  $\mathcal{E}$ -functions which vanish for  $s = 0$ ,







$s = 1$ ; i.e.,  $u_n(s) = c_n \sin n\pi s$  are E-functions, where  $c_n$  is determined such that  $\|u_n\| = 1$ ;  $\mu_n = 1 + n^2\pi^2$  are the E-values of  $M$ . Since the form  $Q$  of this example possesses a positive-discrete spectrum, the expansion theorems of Ch. III and Ch. IV supplement yield for every  $x(s)$  in  $\hat{\mathcal{J}}$   $x(s) = \sum_n \alpha_n c_n \sin n\pi s$ ,

$$\int_0^1 x^2 ds = \sum_n K_n \alpha_n^2, \quad \int_0^1 \{Dx^2 + x^2\} ds = \sum_n \alpha_n^2,$$

and, if  $x$  is in  $\hat{\mathcal{F}}$ ,

$$Mx = \sum_n \alpha_n c_n \sin n^2\pi^2 s, \text{ or}$$

$$-D^2x = \sum_n n^2\pi^2 \alpha_n c_n \sin n^2\pi^2 s.$$

Example 3.  $0 \leq s \leq 1$ ,  $p = s$ ,  $q = n^2 s^{-1} + s$ ,  $r = s$ .

$M = -s^{-1}DsD + n^2 s^{-2} + 1$ . The E-equation

$$s^{-1}DsDu - n^2 s^{-2}u + (\mu - 1)u = 0$$

has for solutions the Bessel functions  $J_n(\tau s)$ ,  $Y_n(\tau s)$  and their linear combinations. The functions  $Y_n$  behave like  $s^{-n}$  for  $n \neq 0$ , like  $\log s$  for  $n = 0$ ; hence  $\|Y_n\| = \infty$ . Therefore, they must be discarded. The argument of Lemma 1 (preceding example) shows that  $J_n(\tau s)$  is also to be discarded unless  $J_n(\tau) = 0$ . i.e., unless  $\tau = \tau_{nm}$  is a root of the Bessel function  $J_n$ . It then must be shown that  $J_n(\tau_{nm} s)$  is in  $\hat{\mathcal{J}}$ . For the endpoint  $s = 1$  the argument of Lemma 2 is to be employed. For  $s = 0$  an even simpler argument is possible since  $J_n(\tau s)$  behaves like  $s^n$  for  $s = 0$ .

the first part of the year, the weather was very dry and the crops were much injured.

the second part of the year, the weather was very wet and the crops were much injured.

the third part of the year, the weather was very dry and the crops were much injured.

the fourth part of the year, the weather was very wet and the crops were much injured.

the fifth part of the year, the weather was very dry and the crops were much injured.

$$x^2 + y^2 = z^2$$

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the sixth part of the year, the weather was very dry and the crops were much injured.

the seventh part of the year, the weather was very wet and the crops were much injured.

the eighth part of the year, the weather was very dry and the crops were much injured.

the ninth part of the year, the weather was very wet and the crops were much injured.

the tenth part of the year, the weather was very dry and the crops were much injured.

the eleventh part of the year, the weather was very wet and the crops were much injured.

the twelfth part of the year, the weather was very dry and the crops were much injured.

the thirteenth part of the year, the weather was very wet and the crops were much injured.

the fourteenth part of the year, the weather was very dry and the crops were much injured.

The case  $n=0$ , however, needs a special treatment.

We replace  $u=J_0(\tau s)$  by

$$\begin{aligned} u_\xi(s) &= u(s) && \text{for } s \geq \xi \\ &= (\log \xi^{-1})^{-1} (\log \xi^{-2} s) u(\xi) && \text{for } \xi^2 \leq s \leq \xi \\ &= 0 && \text{for } 0 < s < \xi^2. \end{aligned}$$

$$\begin{aligned} \text{Then } \int_0^\eta s D(u-u_\xi)^2 ds &\leq 2 \int_0^\xi s D u^2 ds + 2 \int_{\xi^2}^\xi s D u_\xi^2 ds \\ &= 2 \int_0^\xi s D u^2 ds + (\log \xi^{-1})^{-1} u^2(\xi) \rightarrow 0 \text{ since } u(0)=1. \end{aligned}$$

$$\int_0^\eta s (u-u_\xi)^2 ds \leq 2 \int_0^\xi s u^2 ds + 2 \int_0^\xi u_\xi^2 ds \rightarrow 0.$$

This indicates that it is possible to approximate  $J_0(\tau s)$  by functions in  $\mathcal{J}$  if  $J_0(\tau)=0$ .

Since in this example the spectrum is positive discrete, expansions with respect to Bessel-functions are established.

In example 2 we have,  $-1 \leq s \leq 1$ ,  $p=(1-s^2)$ ,  $q=r=1$ .

$M=-D(1-s^2)D+1$ ; the  $E$ -value equation is

$D(1-s^2)Du + (\mu-1)u=0$ . It can be shown (cf. e.g. Courant-Hilbert Vol. I Ch. V § 10) that the solutions of this equation are either Legendre polynomials or behave like  $\log(1-s)$  at  $s=1$ , or like  $\log(1+s)$  at  $s=-1$ . For the latter functions

$\int_{-1}^1 (1-s^2) D u^2 ds = \infty$ ; hence they are excluded. That the Legendre polynomials are in  $\mathcal{J}$  can be shown by the same reasoning that was applied to the Bessel function  $J_0$  in connection with example 3. The Legendre functions thus constitute the  $E$ -functions. Since the spectrum is positive discrete in this case, expansion theorems are proved.



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## CHAPTER V. Spectral Resolution of Operators.

$\mathcal{H}$ : Hilbert space,  $Q$  bounded symmetric linear operator.

$$\|Qx\| \leq K\|x\|, \quad (x, Qx) = (Qx, x).$$

We are seeking a generalization of the concept of E-vector.

Such a generalization is that of E-vectors belonging to an E-interval.

Let  $\Delta K$  be a  $K$ -interval, we consider intervals with and without end points, also a single point is an interval.

In case the discrete spectrum is complete, we say, the E-space  $\Delta \mathcal{H}_K$  to the E-interval  $\Delta K$  is the subspace spanned by the E-vectors to E-value  $K$  in  $\Delta K$ ; E vectors to  $\Delta K$  are vectors in  $\Delta \mathcal{H}_K$ . We observe that  $\Delta \mathcal{H}_K + \Delta \mathcal{H}_L$  if  $\Delta K \cdot \Delta L$  is void. Further  $(u, Qu)(u, u)$  is in  $\Delta K$  for  $u$  in  $\Delta \mathcal{H}_K$ . Our aim is to obtain a characterization of E-spaces  $\Delta \mathcal{H}_K$  without reference to E-values. This can be done by means of the concept of functions of an operator.

First the operator  $Q^n$  can be formed from  $Q$ ,  $n=0,1,2,\dots$ . Further, if  $p(K)$  is a polynomial  $p(K)=a_0+a_1K+\dots+a_nK^n$ , we can define  $p(Q)=a_0+a_1Q+\dots+a_nQ^n$ . We shall see that such functions of the operator  $Q$  can also be defined for larger classes of functions  $f(K)$ .

In case the complete spectrum is complete, the definition of  $f(Q)$  can be based on the spectral representation.  $x=\sum_n \lambda_n u^n$  and  $Qx = \sum_n \lambda_n \lambda_n u^n$ . We see immediately that for polynomial  $p(K)$ , we have  $p(Q)x = \sum_n p(\lambda_n) \lambda_n u^n$ . Hence we may define for any bounded function  $f(K)$ :  $f(Q)x = \sum_n f(\lambda_n) \lambda_n u^n$ .



We have a mapping of the functions  $f(K)$  to operators, retaining all functional relations; in particular

$$f(Q)g(Q) = fg(Q) \quad \text{if } f(K)g(K) = fg(K).$$

i.e. multiplication corresponds to successive application.

Consider the function  $f_*(K)=1$  for  $K=K_*$ ,  $=0$  for  $K \neq K_*$ ;

then  $f_*(Q)x = \sum_{K_n=K_*} a_n u^n$ ; i.e.  $f_*(Q)x$  is the pro-

jection of  $x$  into the  $E$ -subspace  $\mathcal{H}_*$  to  $K_*$ . The vectors to  $K_*$  thus can be characterized as the vectors of the form  $f_*(Q)x$ .

Let  $\Delta_K$  be an  $\alpha$ -interval; consider the function

$f_\Delta(K)=1$  in  $\Delta_K$ ,  $=0$  outside. Then  $f_\Delta(Q)x = \sum_{K_n \in \Delta_K} a_n u^n$ ;

i.e.  $f_\Delta(Q)x$  is the projection of  $x$  into the  $E$ -subspace

$\mathcal{H}_\Delta$  to  $\Delta_K$ , and the vectors in  $\mathcal{H}_\Delta$  can be characterized as those of the form  $f_\Delta(Q)x$ .

By "spectral resolution" we mean from now on, to obtain these operators  $f_\Delta(Q)$ .

It was the important discovery of E. Riesz that for any symmetric bounded operator  $Q$ , a definition of  $f(Q)$  can be given without reference to spectral representation, and that a spectral resolution can be obtained from these operator functions

It is a common mistake to think that the

only way to improve the quality of the

work is to

work longer hours and to

work harder and to work faster

and to work more efficiently

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and to work more innovatively

and to work more imaginatively

and to work more strategically

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and to work more politically

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and to work more divinely



In what follows functions  $f(K)$  are supposed to be defined in  $|K| \leq k$  and all statements refer to this interval. Let  $f(K)$  represent functions of a class containing sum and product. We say, the linear symmetric operator  $f(\lambda)$  is "properly" defined for this class if the following statements hold.

A If  $f(K) = f_1(K) + f_2(K)$  then  $f(\lambda) = f_1(\lambda) + f_2(\lambda)$

B If  $f(K) = f_1(K) \cdot f_2(K)$  then  $f(\lambda) = f_1(\lambda) \cdot f_2(\lambda)$

where in the last formula the  $\cdot$  represents successive application.

C If  $f_1(K) \geq f_2(K)$  then  $(x, f_1(\lambda)x) \geq (x, f_2(\lambda)x)$ .

For the class of polynomials  $p(K)$  the operator  $p(\lambda)$  was defined;  $p(\lambda)$  is obviously linear and symmetric.

We have

**Theorem 5.1** The definition of the operators  $p(\lambda)$  for polynomials  $p(\lambda)$  is proper.

Properties A, B are obvious. It is property C on which the theory of F. Riesz is essentially based. To prove C we refer to a purely algebraic

**Lemma 5.1** If the polynomial  $p(K) \geq 0$  for  $|K| \leq k$ , it can be written as the sum of polynomials of the form  $g(K)j^2(K)$  where  $j(K)$  is

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any polynomial and  $g(K)$  is either 1, or  $k-K$ , or  $k+K$ , or  $k^2 - K^2$ .

Proof: Problem.

From this lemma we derive property C for polynomials  $p(K)$  immediately. Since  $p_1(K) - p_2(K)$  is a polynomial it is sufficient to assume  $p_1(K) = p(K)$ ,  $p_2(K) = 0$ . We set  $j(Q)x = y$ .

Then 1.  $(x, j^2(Q)x) = (y, y) \geq 0$

2.  $(x, (k+Q)j^2(Q)x) = (y, (k+Q)y) = k(y, y) + (y, Qy) \geq 0$

because of  $|y, Qy| \leq \|y\| \|Qy\| \leq k\|y\|^2$ .

3.  $(x, (k^2 - Q^2)j^2(Q)x) = k^2(y, y) - (Qy, Qy) \geq 0$ .

Hence, in view of the lemma,  $p(K) \geq 0$  implies  $(x, p(Q)x) \geq 0$  and theorem 5.1 is proved.

Now let  $f(K)$  be a continuous function (for  $|K| \leq k$ ).

Approximate it, according to Weierstrass, by a sequence of polynomials  $p_n(K)$ , uniformly (in  $|K| \leq k$ ). Set  $p_{mn}(K) = p_m(K) - p_n(K)$ .

To every  $\varepsilon > 0$  there is a  $N(\varepsilon)$  such that  $|p_{mn}(K)| \leq \varepsilon$  (in  $|K| \leq k$ ).

Hence  $p_{mn}^2(K) \leq \varepsilon^2$ , and, in view of C,  $(x, p_{mn}^2(Q)x) \leq \varepsilon^2(x, x)$  or

$$\|p_{mn}(Q)x\| \leq \varepsilon \|x\|$$

Consequently, the vectors  $p_n(Q)x$  form for every  $x$  a Cauchy sequence and, since the space is complete, converge to a limit vector, which we denote by  $f(Q)x$ .

-That  $f(Q)$  is independent of the choice of the approximating polynomials follows from the fact that for two <sup>such</sup> sequences

$p_n^{(1)}, p_n^{(2)}$ , one has

$$\|(p_n^{(1)}(Q) - p_n^{(2)}(Q))x\| \leq \max |p_n^{(1)}(K) - p_n^{(2)}(K)| \|x\| \rightarrow 0.$$



Let  $f$  be a function on  $\mathbb{R}^n$  and let  $\mathcal{F}f$  denote its Fourier transform. Then

$$\mathcal{F}(\mathcal{F}f)(\xi) = f(-\xi) \quad (1.1)$$
$$\mathcal{F}(f(x)e^{ix \cdot \eta}) = \delta(\xi - \eta) \quad (1.2)$$
$$\mathcal{F}(f(x)e^{ix \cdot \eta}) = \delta(\xi - \eta) \quad (1.3)$$

$$\mathcal{F}(f(x)e^{ix \cdot \eta}) = \delta(\xi - \eta) \quad (1.4)$$
$$\mathcal{F}(f(x)e^{ix \cdot \eta}) = \delta(\xi - \eta) \quad (1.5)$$

where  $\delta$  is the Dirac delta function. In view of the above, we have

$$\mathcal{F}(\mathcal{F}f)(\xi) = f(-\xi) \quad (1.6)$$
$$\mathcal{F}(f(x)e^{ix \cdot \eta}) = \delta(\xi - \eta) \quad (1.7)$$

$$\mathcal{F}(f(x)e^{ix \cdot \eta}) = \delta(\xi - \eta) \quad (1.8)$$
$$\mathcal{F}(f(x)e^{ix \cdot \eta}) = \delta(\xi - \eta) \quad (1.9)$$
$$\mathcal{F}(f(x)e^{ix \cdot \eta}) = \delta(\xi - \eta) \quad (1.10)$$

$$\mathcal{F}(f(x)e^{ix \cdot \eta}) = \delta(\xi - \eta) \quad (1.11)$$

where  $\delta$  is the Dirac delta function. In view of the above, we have

$$\mathcal{F}(f(x)e^{ix \cdot \eta}) = \delta(\xi - \eta) \quad (1.12)$$

$$\mathcal{F}(f(x)e^{ix \cdot \eta}) = \delta(\xi - \eta) \quad (1.13)$$



One shows easily that  $f(Q)x$  depends linearly on  $x$ ;  
 i.e. that  $f(Q)$  is a linear operator, and also that  $f(Q)$  is  
 symmetric, since the same holds for the approximating polynomials.  
 Further

Theorem 5.2 The operators  $f(Q)$  for continuous  $f(K)$  are properly  
 defined.

The properties A and C follow immediately from the  
 corresponding properties for  $p_n(Q)$ .

However, we want to define operators  $f(Q)$  also for functions  
 $f(K)$  which are only piecewise continuous. What we really need  
 is that  $f(K)$  is bounded and can be approximated by a non-  
 decreasing sequence of continuous functions, i.e.

$$f_1(K) \leq f_2(K) \leq \dots \longrightarrow f(K) \leq F = \text{const};$$

we call such functions lower semi-continuous. From theorem 5.2C

we have  $(x, f_1(Q)x) \leq (x, f_2(Q)x) \leq \dots \leq F(x, x)$ .

The latter sequence, therefore, converges to a limit value; hence

$0 \leq (x, f_{mn}(Q)x) \rightarrow 0$  for  $m \geq n \rightarrow \infty$ . On applying the Schwarz

inequality to the non-negative form  $f_{mn}(Q)$ , we obtain

$$\begin{aligned} (f_{mn}(Q)x, f_{mn}(Q)x)^2 &\leq (x, f_{mn}(Q)x)(x, f_{mn}^3(Q)x) \\ &\leq (x, f_{mn}(Q)x) F^3(x, x) \longrightarrow 0, \end{aligned}$$

since  $f_{mn}^3(K) \leq f_{mn}^3(K) \leq F^3$ .

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Hence  $f_n(\lambda)x$  is a Cauchy sequence and converges to a limit function which we denote by  $f(\lambda)x$ ;  $f(\lambda)x$  depends linearly on  $x$  and one proves easily

Theorem 5.3 The operators  $f(\lambda)$  are properly defined for lower-semi-continuous  $f(K)$ .

Evidently  $f(\lambda)$  is also properly defined for differences of lower-semi-continuous functions, and that is the class of functions we want. For a function that is 1 in  $\Delta K$ , 0 outside is such a difference.

Hence to every interval  $\Delta K$  an operator  $f_{\Delta}(\lambda) = \Delta P$  is defined. It has the following properties:

If  $\Delta K = \Delta_1 K + \Delta_2 K$  then  $\Delta_1 P + \Delta_2 P = \Delta P$ .

$\Delta K = \Delta_1 K \cdot \Delta_2 K$  then  $\Delta_1 P \Delta_2 P = \Delta P$ .

This implies in particular: If  $\Delta_1 K \cdot \Delta_2 K$  is void, then

$\Delta_1 P \Delta_2 P = 0$ ; further  $(\Delta P)^2 = \Delta P$ .

If  $F_1 \leq f(K) \leq F_2$  for  $K$  in  $\Delta K$  then

$$F_1 \leq (f(\lambda)\Delta Px, x)/(\Delta Px, x) \leq F_2.$$

This holds in particular for  $f(K) = K$  and gives

$$(Q\Delta Px, x)/(\Delta Px, x) \text{ is in } \Delta K.$$

To understand the significance of these properties we introduce

the space  $\Delta \mathcal{H}$  of all  $u$  of the form  $u = \Delta Px$ . Since  $f_{\Delta}^2 = f_{\Delta}$  we have  $\Delta P^2 = \Delta P$ ,

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... 1972, p. 100, 101

Source: *Journal of the American Statistical Association*, 1990, 85, 103-111.

[illegible]

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$$f(x) = \frac{1}{2} \left( \frac{1}{x} + \frac{1}{x^2} \right) \quad \text{for } x \in \mathbb{R} \setminus \{0\}$$

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*Journal of Management Studies*, 19(1), 67-80.

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$$(\Delta P y, (1 - \Delta P)x) = (y, (\Delta P - \Delta P^2)x) = 0; \text{ i.e.}$$

$(1 - \Delta P)x \perp \Delta h_y$ . Hence  $x$  can be split up into

$x = \Delta P x + (1 - \Delta P)x$ , where  $\Delta P x$  is in  $\Delta h_y$  and

$(1 - \Delta P)x \perp \Delta h_y$ . This shows:

The operator  $\Delta P$  is the projection into  $\Delta h_y$ .

Let the interval  $|K| \leq k$  be split into a sum of intervals

$\Delta_\mu K$  without common points; then we have

$$\sum_\mu \Delta_\mu P = 1$$

which means that every vector  $x$  can be split up into a sum of

$E$ -vectors

$$\sum_\mu \Delta_\mu P x = x.$$

Let  $K_\mu$  be any value of  $K$  in  $\Delta_\mu K$ . Then the operator

$\sum_\mu K_\mu \Delta_\mu P$  is approximately  $Q$  and

$\sum_\mu f(K_\mu) \Delta_\mu P$  is approximately  $f(Q)$  for continuous  $f$ . The

sums on the left hand side really converge to  $Q$  or  $f(Q)$  respectively if the subdivision is refined.

Consider two subdivisions  $\Delta_\mu$  and  $\Delta'_\nu$ . Then, in view of 1. and

2.,

$$\begin{aligned} & \| \sum_\mu f(K_\mu) \Delta_\mu P x - \sum_\nu f(K'_\nu) \Delta'_\nu P x \|^2 \\ &= \sum_{\mu, \nu} |f(K_\mu) - f(K'_\nu)|^2 \| \Delta_\mu P \Delta'_\nu P x \|^2 \\ &\leq \varepsilon^2 \sum_{\mu, \nu} \| \Delta_\mu P \Delta'_\nu P x \|^2 = \varepsilon^2 \| x \|^2, \end{aligned}$$

1. The first part of the paper is devoted to a study of the

properties of the function  $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  and its

derivatives. It is shown that the function  $f(x)$  is

continuous and differentiable for all values of  $x$ .

2. In the second part of the paper, we consider the

problem of finding the maximum and minimum values of the

function  $f(x)$  on the interval  $[0, 1]$ . It is shown that

the function  $f(x)$  has a maximum value at  $x = 1$  and a

minimum value at  $x = 0$ . The maximum value is  $e$  and the

minimum value is 1.

3. In the third part of the paper, we consider the

problem of finding the maximum and minimum values of the

function  $f(x)$  on the interval  $[0, 1]$ . It is shown that

the function  $f(x)$  has a maximum value at  $x = 1$  and a

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4. In the fourth part of the paper, we consider the

problem of finding the maximum and minimum values of the

function  $f(x)$  on the interval  $[0, 1]$ . It is shown that

the function  $f(x)$  has a maximum value at  $x = 1$  and a

where  $\varepsilon = \max |f(K_\mu) - f(K'_\nu)|$  for all  $\mu, \nu$  such that  $\Delta_\mu K$  and  $\Delta'_\nu K$  have a common point; and  $\varepsilon$  can be made arbitrarily small.

It is natural to express these facts symbolically by

$$\begin{aligned}\int dP &= 1 \\ \int K dP &= Q \\ \int f(K) dP &= f(Q), \text{ (containing the preceding ones),}\end{aligned}$$

which gives a pleasant form of the spectral resolution. In case of a complete discrete spectrum these integrals degenerate into sums.

We are now called to exemplify the preceding results with at least one example. Take the space  $\mathcal{L}$  of functions  $x(K)$  defined in  $S$ , introduce a unit-form

$$(x, x) = \int_S x^2(K) r(K) dK,$$

and close it off to a space  $\hat{\mathcal{L}}$ . In  $\mathcal{L}$  introduce the operator  $Q$  by

$$Qx(K) = Kx(K).$$

$Q$  is bounded if  $S$  is bounded, which we shall assume although that is not essential. Then  $\mathcal{L}$  can be extended to  $\hat{\mathcal{L}}$ . The functions  $x(K)$  in what follows are assumed to be in  $\mathcal{L}$ . The

Let  $f(x) = x^2 + 1$  and  $g(x) = x^2 - 1$ .  
Then  $f(x) + g(x) = 2x^2$  and  $f(x) - g(x) = 2$ .

It is required to show that  $f(x) + g(x)$  and  $f(x) - g(x)$  are both even functions.

$$\begin{aligned} f(x) + g(x) &= 2x^2 \\ f(x) - g(x) &= 2 \end{aligned}$$

$$\begin{aligned} (f+g)(x) &= 2x^2 \\ (f-g)(x) &= 2 \end{aligned}$$

Since  $2x^2$  and  $2$  are both even functions, it follows that  $f(x) + g(x)$  and  $f(x) - g(x)$  are both even functions.

Q.E.D.

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results then extend immediately to  $x(K)$  in  $\mathcal{L}$ .

For polynomials  $p(K)$  we have

$$p(Q)x(K) = p(K)x(K).$$

For continuous functions  $f(K)$  we have

$$f(Q)x(K) = f(K)x(K).$$

For the functions  $f_{\Delta}(K) = 1$  in  $\Delta K$ ,  $= 0$  outside, we have

$f_{\Delta}(Q)x(K) = f_{\Delta}(K)x(K)$ , which function may be only piecewise continuous and belongs to  $\mathcal{L}$  as such. Hence the projection

$f_{\Delta}(Q) = \Delta P$  is this: from  $x(K)$  it cuts off the part outside  $\Delta K$ .

The  $E$ -functions of the interval  $\Delta K$  are the functions vanishing outside  $\Delta K$ .

The above example permits various generalizations. Instead of  $r(K)dK$  we introduce a non-decreasing function  $\rho(K)$  and replace  $r(K)dk$  by  $d\rho$ ; i.e., we set

$$(x, x) = \int_S x^2(K) d\rho(K).$$

We then even admit discontinuous functions  $\rho(K)$ . What this implies may be seen in the case that  $\rho(K)$  is piecewise constant with jumps  $d\rho_1, d\rho_2, \dots$  at  $K = K_1, K_2, \dots$ .

We have then

$$(x, x) = \sum_{\mu} x_{\mu}^2 d\rho_{\mu}, \quad \text{where } x_{\mu} = x(K_{\mu}),$$

$$(x, Qx) = \sum_{\mu} K_{\mu} x_{\mu}^2 d\rho_{\mu}.$$

Let  $\mathcal{A}$  be a subalgebra of  $\mathcal{B}$ .

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$$\mathcal{A} \cap \mathcal{B} = \mathcal{A}$$

That is, we come back to the case of a pure point spectrum.

If  $\rho(k)$  has discontinuities but is not constant in between we have a combination of continuous and point spectrum.

We may generalize in the following way. We consider as element of the Hilbert space a system of functions

$x = \{x_1(k), x_2(k), \dots\}$ , and take for unit form

$$(x, x) = \int_S x_1^2(k) d\rho_1(k) + \int_S x_2^2(k) d\rho_2(k) + \dots$$

Then

$$Qx = \{Kx_1(k), Kx_2(k), \dots\} \quad \text{and}$$

$$(x, Qx) = \int_S Kx_1^2(k) dk + \dots$$

It can be shown that the latter is the more general case in the following sense:

The Hilbert space can be represented by the space of systems of functions  $\{x_\mu(k)$  such that  $Qx$  is represented by  $\{Kx_\mu(k)\}$  and  $(x, x) = \sum_\mu \int_S x_\mu^2(k) d\rho_\mu(k)$ . This statement implies the general theory of spectral representation.

It is not difficult to indicate how this representation can be achieved. Take any vector  $x_1$  and all vectors  $E_1(Q)x_1$ ; close this subspace off to a space  $\mathcal{H}_{x_1}$ ; we set

$$\Delta \rho_1(k) = (x_1, \Delta P x_1) \quad \text{and find}$$

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$$(x, x) = \sum_{\mu} (x_1, \Delta_{\mu} P x_1) \quad \text{and}$$

$$(f_1(Q)x_1, f_1(Q)x_1) = \lim \sum_{\mu} f_1^2(K_{\mu}) (x, \Delta_{\mu} P x_1)$$

which can be shown to lead to

$$(f_1(Q)x_1, f_1(Q)x_1) = \int f_1^2(K) d\rho_1(K).$$

Hence for every  $x$  in  $\mathcal{H}_1$ , represented by  $f_1(K)$ , we have

$$(x, x) = \int f_1^2(K) d\rho_1(K).$$

We now take the space perpendicular to  $\mathcal{H}_1$ , perform the same operation and obtain a space  $\mathcal{H}_2$ . Continuing, we obtain a sequence of spaces  $\mathcal{H}_1, \mathcal{H}_2, \dots$  which span the whole space  $\mathcal{H}$ , as can be shown (if  $\mathcal{H}$  has countable dimension). Since every  $x$  in  $\mathcal{H}$  can be split into  $x = x_1 + x_2 + \dots$ , with  $x_1$  in  $\mathcal{H}_1$ ,  $x_2$  in  $\mathcal{H}_2$  and so on, this leads to the above spectral representation with

$$x_{\mu}(K) = f_{\mu}(K).$$

(1)  $\frac{1}{x^2} = x^{-2}$

(2)  $\frac{1}{x^3} = x^{-3}$

...

(3)  $\frac{1}{x^4} = x^{-4}$

...

(4)  $\frac{1}{x^5} = x^{-5}$

...

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$\frac{1}{x^6} = x^{-6}$









